ACCELERATION OF MOLECULAR-TRANSPORT PROCESSES IN TURBULENT FLOWS

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Abstract

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V. A. FROST

ACCELERATION OF MOLECULAR-TRANSPORT PROCESSES IN TURBULENT FLOWS

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In describing turbulence in Eulerian variables $y \in R_3$, τ, we do not follow the changes in the parameters of individual fluid particles and therefore cannot properly take into account molecular-transport processes. This is connected with the fact that molecular-transport processes occur between neighboring fluid elements, whereas in an Eulerian description of turbulence it is very difficult to introduce the concept of proximity of fluid particles.

It is more natural to consider molecular-transport processes in turbulent flows in Lagrangian variables $x \in R_3$, τ ([1,3]). However, in problems that are stationary on the average, when it is necessary to satisfy one or another boundary condition, it seems convenient to us to introduce the following coordinates: x are the coordinates of a certain surface on which the fluid particles are labeled; for this it is necessary that the entire fluid flux pass through this surface (below, everywhere, for simplicity, we shall assume $x \in R_2$ and $x = (y_2, y_3)$ for $y_1 = 0$); τ is the instant of observation; ϑ is the time of motion of a particle from the surface x. If τ_0 is the instant at which the particle crosses the surface x, then $\vartheta = \tau - \tau_0$.

In Lagrangian variables the equations describing the motion of an incompressible fluid have the form ([1,3]):

$$\partial y(x,\tau)/\partial\tau = u(x,\tau); \tag{1}$$

$$\partial u(x,\tau)/\partial\tau = \mathrm{Re}^{-1}\,\Delta_y u(x,\tau) - \operatorname{grad}\pi(x,\tau); \tag{2}$$

$$|D(y)/D(x)| = 1, \tag{3}$$

where $\pi(x,\tau)$ is the pressure; Δ_y denotes the Laplacian with respect to the variables y. In the variables x, ϑ, τ the equations of motion will be:

$$\partial y(x,\vartheta,\tau)/\partial\tau + \partial y(x,\vartheta,\tau)/\partial\vartheta = u(x,\vartheta,\tau); \tag{4}$$

$$\partial u(x,\vartheta,\tau)/\partial\tau + \partial u(x,\vartheta,\tau)/\partial\vartheta = \mathrm{Re}^{-1}\,\Delta_y u - \mathrm{grad}\,\pi(x,\vartheta,\tau); \tag{5}$$

$$|D(y)/D(x,\vartheta)| = u_1(x,0,\tau-\vartheta). \tag{6}$$

Analogously, the transport equation for some substance φ (temperature or concentration)

$$\partial\varphi(x,\vartheta,\tau)/\partial\tau + \partial\varphi(x,\vartheta,\tau)/\partial\vartheta = \mathrm{Fo}\,\Delta_y\varphi(x,\vartheta,\tau), \tag{7}$$

where Fo is a certain dimensionless criterion; if φ is temperature, then Fo is the well-known Fourier criterion.

We shall regard the turbulent velocity field as given and calculate certain statistical properties of the transport field $y(x,\vartheta,\tau)$ needed for computing the field $\varphi(x,\vartheta,\tau)$. For simplicity we shall neglect fluctuations of φ.

By the Friedman–Keller method we obtain equations describing the statistical properties of the fields $y(x,\vartheta,\tau)$ and $\varphi(x,\vartheta,\tau)$. Averaging (4) and (7), we obtain

$$\partial\langle y(x,\vartheta,\tau)\rangle_\tau/\partial\vartheta = \langle u(x,\vartheta,\tau)\rangle_\tau; \tag{8}$$

$$\partial\langle\varphi(x,\vartheta,\tau)\rangle_\tau/\partial\vartheta = \langle\Delta_y\varphi\rangle_\tau. \tag{9}$$

Multiplying (4) by y_i and averaging, we obtain

$$\partial\langle y_i(x,\vartheta,\tau)y_j(x',\vartheta',\tau')\rangle_\tau/\partial\vartheta = \langle y_j(x',\vartheta',\tau')u_i(x,\vartheta,\tau)\rangle_\tau; \tag{10}$$

$$\partial\langle u_i(x,\vartheta,\tau)y_j(x',\vartheta',\tau')\rangle_\tau/\partial\vartheta = \langle u_i(x,\vartheta,\tau)u_j(x',\vartheta',\tau')\rangle_\tau, \tag{11}$$

where $\langle\ \rangle_\tau$ denotes averaging with respect to τ. Let us denote

$$\begin{aligned}\langle y_i y_j\rangle &= K_{ij}(\vartheta,\vartheta';x,x'),\\ \langle y_i u_j\rangle &= L_{ij}(\vartheta,\vartheta';x,x'),\\ \langle u_i u_j\rangle &= M_{ij}(\vartheta,\vartheta',x,x').\end{aligned} \tag{12}$$

Let us integrate (10) and (11) under the condition that

$$L_{ij}(0, \vartheta'; x, x') = K_{ij}(0, \vartheta'; x, x') = K_{ij}(\vartheta, 0, x, x') = 0.$$

This means that displacements of particles in the plane $y_1 = 0$ are absent. Then

$$K_{ij}(\vartheta, \vartheta'; x, x') = \int_0^{\vartheta} \int_0^{\vartheta'} M_{ij}(\vartheta'', \vartheta'''; x, x') \, d\vartheta''' \, d\vartheta'', \tag{13}$$

which in form must coincide with the expression for turbulent transport in Lagrangian coordinates [2].

Under the assumption of homogeneity of the turbulent velocity field,

$$M_{ij}(\vartheta, \vartheta', x, x) = M_{ij}(|\vartheta - \vartheta'|), \tag{14}$$

$$K_{ij}(\vartheta, \vartheta'; x, x) = 2 \int_0^{\vartheta} (\vartheta - p) M_{ij}(p) \, dp = \mathrm{invar}(x). \tag{15}$$

Let us note that, knowing the correlation function $M_{ij}(p)$, one can write an expression for the Eulerian correlation functions and attempt to obtain the properties of the correlation functions M_{ij}. For example,

$$M_{ij}(0) = \langle u_i u_j \rangle = R_{ij}(0),$$

where R_{ij} is the Eulerian correlation function.

Consider equation (3). We must transform the Laplacian with respect to the variables y to our variables, making use of the form of representation through the Jacobian (6) [1]; for the plane case we obtain

$$\Delta_y \varphi = \left(u^2(x, 0, \tau)\right)^{-1} \left\{ \left[(\partial y_1/\partial \vartheta)^2 + (\partial y_2/\partial \vartheta)^2 \right] \partial^2 \varphi/\partial x_1^2 + \left[(\partial y_1/\partial x_1)^2 + (\partial y_2/\partial x_1)^2 \right] \partial^2 \varphi/\partial \vartheta^2 + A \partial^2 \varphi/\partial x_1 \partial \tag{16}$$

If, as was indicated above, we confine ourselves only to the mean rate of change and do not consider the fluctuations of φ, then

$$\partial \langle \varphi(x, \vartheta, \tau) \rangle_\tau / \partial \vartheta = \mathrm{Fo} \, \langle u_1^2 \rangle^{-1} \left\{ \left[\overline{(\partial y_1/\partial \vartheta)^2} + \overline{(\partial y_2/\partial \vartheta)^2} \right] \partial^2 \langle \varphi \rangle / \partial x_1^2 + \left[\overline{(\partial y_1/\partial x_1)^2} + \overline{(\partial y_2/\partial x_1)^2} \right] \partial^2 \langle \varphi \rangle / \partial \vartheta^2 \right\} \tag{17}$$

for homogeneous turbulence the remaining terms of equation (16) vanish.

Taking into account (14), (15) and the rules for obtaining averages from correlation functions, we obtain

$$\frac{\partial\langle\varphi\rangle}{\partial\vartheta} = F_0 \left\{ \left(1 + \frac{\langle u_i u_i\rangle}{\langle u_1^2\rangle}\right) \frac{\partial^2\langle\varphi\rangle}{\partial x_1^2} + 2\langle u\rangle^{-2} \left(1 + \int_0^\vartheta (\vartheta - p)\, \frac{\partial^2 M_{ii}(p,x,x')}{\partial x_1\, \partial x_1'}\bigg|_{x_1=x_1'} dp\right) \frac{\partial^2\langle\varphi\rangle}{\partial\vartheta^2} \right\} + W(\varphi), \tag{18}$$

to which we have added a source function $W(\varphi)$, describing possible chemical reactions and combustion.

The first term contains a factor allowing for the fact that, in a turbulent flow, a material line has, on the average, nonzero curvature; however, the influence of this factor is very small. The influence of the second term manifests itself in a considerably more complicated way. It is possible that

$$\int_0^\vartheta (\vartheta - p)\, \frac{\partial^2 M_{ii}(p,x,x')}{\partial x_1\, \partial x_1'}\bigg|_{x_1=x_1'} dp$$

behaves analogously to the mean-square displacement in Taylor’ s theory, i.e., for small ϑ it is $\sim \vartheta^2$, while for large ϑ it is $\sim \vartheta$ ([2]). One of the consequences of this is that there is no combustion velocity, constant in ϑ, in a plane turbulent flame front. The combustion velocity increases progressively.

Let us obtain approximate formulas for the field $\langle\varphi\rangle = \varphi(y)$ in Eulerian variables.

$$\overline{\varphi}(y) = \int_0^\infty \int_x \langle\varphi(x,\vartheta,\tau)\rangle_\tau\, p(y,x,\vartheta)\, dx\, d\vartheta, \tag{19}$$

where $p(y,x,\vartheta)$ is the probability density for a particle (x,ϑ) to arrive at the point y. For an exact calculation of the function $p(y,x,\vartheta)$, it is necessary to know an infinite set of moments of all possible orders of the function $y(x,\vartheta,\tau)$. As a first approximation one may regard the transport field as a normal random field, for whose description the already known second-order moments are sufficient. Then

$$\overline{\varphi}(y) \simeq N \int_0^\infty \int_x \frac{\langle\varphi(x,\vartheta,\tau)\rangle_\tau}{\sqrt{(2\pi)^3 K_{11} K_{22}}} \exp\left(-\sum_{i=1}^{2} \frac{\left(y_i - \langle y_i(x,\vartheta)\rangle_\tau\right)^2}{2K_{ii}(\vartheta,\vartheta;\, x,x)}\right) dx\, d\vartheta; \tag{20}$$

because of the approximate nature of the formula, introduction of the normalizing factor N is necessary.

Under the same assumptions one can obtain a formula for $p(\varphi,y)d\varphi$, the probability of encountering at the given point y, $\varphi \in (\varphi, \varphi + d\varphi)$,

$$p(\varphi, y) = N \int \frac{1}{\sqrt{(2\pi)K_{11}K_{22}}} \exp\left(-\sum_{i=1}^{2} \frac{\left(y_i - \langle y_i(x, \vartheta, \tau)\rangle_\tau\right)^2}{2K_{ii}(\vartheta, \vartheta; x, x)} \right) d\vartheta, \quad (21)$$

where the integration is carried out along the line $x = x(\vartheta)$, along which $\langle \varphi(x, \vartheta, \tau)\rangle_\tau = \varphi$. In conclusion, we note that if in the space (x, ϑ) there exist regions in which $\langle \varphi(x, \vartheta, \tau)\rangle_t = \varphi = \text{const}$, then for these φ there is no probability density $p(\varphi, y)$. In this case one should consider $P(\varphi', y)$—the probability of encountering at the given point y, $\varphi < \varphi'$.

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Note: Figure translations are in progress. See original paper for figures.