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A GENERAL SINGULAR EQUATION ON AN OPEN CONTOUR AND A GENERALIZED ABEL EQUATION

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A large class of functional equations is known, the so-called singular equations (with a Cauchy kernel, convolution type, etc.), which have a common theory —the Noether theory ([1−5]). The common character of these equations follows from the fact that a certain part of the equation, called characteristic, is reduced to the Riemann boundary-value problem and determines the index of the equation. A number of works ([6−11]) and others are devoted to the general theory of singular equations. For a singular equation in a Banach space, the qualitative aspect—the Noether property of the equation—has been completely studied. The constructive aspect—the solution of the characteristic equation and the effective determination of the index by means of the most general theory—was considered only by Yu. I. Cherskii ([9]). In ([9]) a singular equation is studied on the basis of the solution of the Riemann problem in a Banach space. In ([11]) the theory ([9]) is extended to an exceptional case. The abstract scheme proposed by Yu. I. Cherskii, however, is applicable only to equations given on a closed contour. In the present article a generalization of the results of Yu. I. Cherskii ([9]) is proposed, making it possible to include in the general theory all concrete types of equations studied so far, including equations with a Cauchy kernel on an open contour and with discontinuous coefficients.

Two methods are known for solving the Riemann problem on an open contour Γ. One ([2]) consists in reducing it to the case of a closed contour Γ_0; the other ([1]) gives an independent solution on the basis of the properties of the integral with Cauchy kernel. We construct a general theory proceeding from the first method. The difficulties that arise here are connected with the consideration of several spaces E, E_1, X, X_1. E and X correspond to classes of functions defined on Γ and Γ_0, while E and E_1 (X and X_1) correspond to different behavior of functions at the endpoints of Γ. For clarity of the abstract scheme, in No. 5 an interpretation is given of the operators and spaces for an equation with Cauchy kernel. In No. 6 a realization of the theory for the generalized Abel equation is indicated.

No. 1. Everywhere below, a symbol of the type $[Y \to Z]$ denotes the ring of linear continuous operators from Y into Z.

Definition 1. E is a subspace of the Banach space X such that there exists an operator $P \in [X \to E]$, identical on E.

Since E is closed in X, without loss of generality one may identify E^* with the space of functionals $\bar{\varphi}$ such that: 1) $\bar{\varphi} \in X^*$; 2) $(\bar{\varphi}, P\varphi) = (\bar{\varphi}, \varphi)$ for $\varphi \in X$.

Theorem 1. E^* is a subspace of X^*; $P^* \in [X^* \to E^*]$ and $P^*\bar{\varphi} = \bar{\varphi}$ for $\bar{\varphi} \in E^*$.

Definition 2. An operator S is called **singular** if $S \in [X \to X]$ and $S^2 = I$, $S \neq \pm I$, where I is the identity operator. The **ring** R is the ring of operators A such that

$$R \subset [E \to E] \cap [X \to X],$$

$PA = AP$, and the operator $P(SA - AS)$ is regular ([6,9]) in E. The equation

$$A_1\varphi + A_2PS\varphi + T\varphi = f, \tag{1}$$

where $\varphi \in E$, $f \in E$, $A_1 \in R$, $A_2 \in R$, T is regular, and S is singular operators, is called a general singular equation (g.s.e.) (cf. (9)). The equation

$$A_1^*\bar{\varphi} + P^*S^*A_2^*\bar{\varphi} + T^*\bar{\varphi} = \bar{f}, \qquad \bar{f} \in E^*, \quad \bar{\varphi} \in \bar{E}^*, \tag{2}$$

is adjoint to (1).

We consider the normal case $(A_1 \pm A_2P)^{-1} \in R$.

No. 2. Following (9), let us consider an analogue of the Riemann problem. Let

$$E^{\pm} = (I \pm S)(E), \qquad \bar{E}^{\pm} = (I \mp S^*)(E^*).$$

Definition 3. Let $\Psi_{\pm}$ be operators mapping X into a Banach space X_1 such that $S \in [X_1 \to X_1]$. Let E_1 be a subspace in X_1. Denote $E_1^{\pm} = (I \pm S)(E_1)$, $X_1^{\pm} = (I \pm S)(X_1)$. We assume that the following properties hold for $\Psi_{\pm}$:
1) $\Psi_{\pm} \in [E^{\pm} \to E_1^{\pm}] \cap [E \to E_1]$;
2) $\Psi_{\pm}^{-1} \in [E_1^{\pm} \cap \Psi_+(X) \to E^{\pm}] \cap [E_1 \cap \Psi_+(X) \to E]$;
3) $\Psi_+(X)$ is invariant with respect to S. Let now U be an operator with the properties:
1) $U \in [E^+ \to E^+] \cap [E_1^+ \to E_1^+] \cap [E \to E] \cap [E_1 \to E_1]$;
2) $U^{-1} \in [E^- \to E^-] \cap [E_1^- \to E_1^-]$;
3) there exists in E_1^+ $(\bar{E}^-)$ an element $h_+ \neq 0$ $(\bar{h}_- \neq 0)$, unique up to a constant factor, such that $U^{-1}h_+ \in E_1^-$ $(U^*\bar{h}_- \in \bar{E}^+)$;
4) $\Psi_+(X)$ is invariant with respect to U, U^{-1}.

Definition 4. Ind $A = \varkappa$, if the representation

$$\Psi_+ A = U^{\varkappa} \Psi_- \tag{3}$$

holds.

The Riemann problem consists in finding elements $\varphi_\pm \in E_\pm$ subject to the condition

$$\varphi_+ = A\varphi_- + g, \qquad g \in E, \quad \operatorname{Ind} A = \varkappa. \tag{4}$$

Theorem 5 from (9), proved for $E = X = X_1 = E_1$, carries over without changes to our case: if $\varphi_+ = U^{\varkappa}\varphi_- + f_-$, $\varphi_\pm \in X_1^\pm$, $f_- \in X_1^-$,

$$\varkappa \geq 0, \quad \text{then} \quad \varphi_+ = \sum_{k=0}^{\varkappa-1} c_k U^k h_+ = P_{\varkappa-1} h_+;$$

the elements $h_+, \dots, U^k h_+$ are linearly independent.

Corollary. $U^{n-1}h_+ \in \Psi_+(E^+)$, $U^{-n}h_+ \in \Psi_-(E^-)$, $n > 0$.

For $\varkappa \geq 0$, (4) is solved by applying the indicated theorem and corollary. For $\varkappa < 0$, for the solvability of (4) it is necessary and sufficient that

$$(\bar{h}_-, U^k \Psi_+ g) = 0, \qquad k = 1, 2, \dots, -\varkappa.$$

No. 3. Setting $\varphi = \varphi_+ - \varphi_-$, $\varphi_\pm \in E_\pm$, we reduce (1), for $T = 0$, to (4), where $A = (A_1 + A_2 P)^{-1}(A_1 - A_2 P)$, $g = (A_1 + A_2 P)^{-1} f$. Solving (4) under the assumption that $\varkappa \geq 0$, we have

$$\varphi = (A_1 - A_2 P)^{-1}\{A_2 P \Psi_+^{-1} P_{\varkappa-1} h_+ + (A_1 - A_2 \Psi_+^{-1} P S \Psi_+)(A_1 + A_2 P)^{-1} f\}. \tag{5}$$

Putting $A_2^* \bar{\varphi} = \bar{\varphi}_- - \bar{\varphi}_+$, $\bar{\varphi}_\mp \in \bar{E}^\mp$, from (2), for $T = 0$, we obtain

$$(A_1^* - P^* A_2^*)(A_1^* + P^* A_2^*)^{-1}(\bar{f} - P^* \bar{\varphi}_+) = \bar{f} - 2P^* \bar{\varphi}_-.$$

Since $P^* \bar{\varphi}_- = \bar{\varphi}_- + (P^* - I)\bar{\varphi}_+$, it follows that $\bar{\varphi}_- = A^* \bar{\varphi}_+ + \bar{g}$, $\bar{g} = A_2^*(A_1^* + P^* A_2^*)^{-1} \bar{f}$. In solving this problem, one uses

Theorem 2. If $N_\pm \in [E^\pm \to E_1^\pm] \cap [E \to E_1]$, then $N_\pm^* \in [\bar{E}_1^\pm \to \bar{E}^\pm] \cap [E_1^* \to E^*]$.

Corollary. $\operatorname{Ind} A^* = -\operatorname{Ind} A$.

No. 4. **Theorem 3.** The general form of a regularizer, left and right, for the g.s.e. (1) is as follows:

$$M' = (A_1 + A_2 P)^{-1} A_1 (A_1 - A_2 P)^{-1} -$$

$$-(A_1 + A_2 P)^{-1} A_2 (A_1 - A_2 P)^{-1} PS + T'. \qquad (6)$$

M' effects an equivalent, for any right-hand side, regularization on the left (on the right) if and only if $\varkappa \geq 0$ ($\varkappa \leq 0$).

The validity of Noether' s theorems for equation (1) is not difficult to derive, on the basis of F. V. Atkinson' s results (7), from the existence of the regularizer (6).

No. 5. For the equation

$$a_1(t)\varphi(t) + \frac{a_2(t)}{\pi i} \int_\Gamma \varphi(\tau)(\tau - t)^{-1}\, d\tau + T\varphi = f(t)$$

we denote: Γ_0 is a closed contour containing Γ; $p(t)$ is the characteristic function of the contour Γ; $A(t) = (a_1 - a_2 p)(a_1 + a_2 p)^{-1}$, and introduce, according to $A(t)$, the weight functions $\rho(t)$ and $\rho_1(t)$, as is done in (3) (ρ_1 corresponds to the assumption of singularities at all discontinuity points of $A(t)$). Let, moreover, $\omega^-(t)/\omega^+(t)$ be a multiplier eliminating the discontinuities of $A(t)$ (2). Then

$$A_j\varphi \equiv a_j(t)\varphi(t), \quad j = 1, 2; \qquad P\varphi \equiv p(t)\varphi(t); \qquad S\varphi \equiv \frac{1}{\pi i} \int_{\Gamma_0} \varphi(\tau)(\tau - t)^{-1}\, d\tau$$

$$X = \mathcal{L}_p(\rho; \Gamma_0), \qquad X_1 = \mathcal{L}_p(\rho_1; \Gamma_0), \qquad E = \mathcal{L}_p^0(\rho; \Gamma), \qquad E_1 = \mathcal{L}_p(\rho_1; \Gamma) *;$$

$$U\varphi \equiv (t - z_0)\varphi(t), \qquad z_0 \in D^+; \qquad \Psi_\pm \varphi \equiv (\omega^\pm X^\pm)^{-1}\varphi(t); \qquad p > 1,$$

where $X^\pm(t)$ are the limiting values of analytic functions that do not vanish anywhere.

No. 6. The results of Nos. 1-4 are applicable to the equation in integrals of fractional order (the generalized Abel equation):

$$u(x) \int_a^x \frac{\varphi(t)}{(x - t)^\mu}\, dt + v(x) \int_x^b \frac{\varphi(t)}{(t - x)^\mu}\, dt + \int_a^b K(x, t)\varphi(t)\, dt = f(x), \qquad a \leq x \leq b, \qquad (7)$$

and to its adjoint

$$\int_a^x \frac{v(t)\psi(t)}{(x-t)^\mu}\,dt + \int_x^b \frac{u(t)\psi(t)}{(t-x)^\mu}\,dt + \int_a^b K(t,x)\psi(t)\,dt = g(x), \qquad a \le x \le b, \ (8)$$

where $0 < \mu < 1$, $K(x,t) = c(x,t)|x-t|^{-\lambda}$, $\lambda < \mu$, $|\partial c(x,t)/\partial t| < \text{const}/|x-t|$, and $u(x)$ and $v(x)$ are continuous functions. Equation (6) for $K(x,t) \equiv 0$ was first studied by K. D. Satsaliuk (12).

Let Γ_0 be a closed contour containing $[a,b]$. Denote

$$F\varphi = \int_a^t \varphi(\tau)(t-\tau)^{-\mu}\,d\tau, \quad t \in \Gamma_0; \qquad r_b = (b-x)^{1-\mu}, \qquad S_{\Gamma_0}\varphi = \frac{1}{\pi i}\int_\Gamma \varphi(\tau)(\tau-t)^{-1}\,d\tau.$$

Then for (7) we have:

$$S\varphi \equiv F^{-1} r_b S_{\Gamma_0} \frac{1}{r_b} F\varphi, \qquad P\varphi \equiv F^{-1} p(\tau) F\varphi.$$

We seek solutions of equation (8) in the class of functions $\psi \in \mathcal{L}_q(\rho_0^{1-q}, [a,b])$, where $\rho_0 = \rho r_b^{-p}$, ρ is introduced from the function $A(t) = (u - ve^{\mu\pi i})(u - ve^{-\mu\pi i})^{-1}$ analogously to No. 5, and solutions of equation (7) in the class of generalized functions φ such that $\varphi = \Phi^{(1-\mu)}$, $\Phi \in \mathcal{L}_p(\rho_0, [a,b])$, $pq = p+q$. The index of equation (7) is $\varkappa = [\theta(b)/2\pi] + 1$, where $\theta(b) = \arg A(b)$. Take $f \in \mathcal{L}_p(\rho_0; [a,b])$ and $g^{(1-\mu)} \in \mathcal{L}_1(\rho_0^{1-q}, [a,b])$. Suppose that $u^2 + v^2 - 2uv\cos\mu\pi = 1$. Then, for $K(x,t) \equiv 0$, from (5) we obtain, in closed form, the solution of equation (7):

$$\varphi = \frac{d}{dx}\int_a^x (x-t)^{\mu-1}\left\{(b-t)^{1-\mu} v(t) Z(t) P_{\varkappa-1}(t) + \right.$$

$$+ \frac{\sin\mu\pi}{\pi}\left[u(t) - 2\cos(\mu\pi)v(t)\right] f(t) -$$

$$\left. - \left(\frac{\sin\mu\pi}{\pi}\right)^2 v(t) Z(t) \int_t^b (\tau-t)^{-\mu}\left(\frac{d}{d\tau}\int_a^\tau \frac{f(w)}{Z(w)}(\tau-w)^{\mu-1}\,dw\right) d\tau\right\} dt,$$

* The index 0 means that the functions in $\mathcal{L}_p^0(\rho;\Gamma)$ are identically equal to 0 outside Γ.

where $P_{\varkappa-1} = \sum_{k=0}^{\varkappa-1} c_k t^k$, the c_k are arbitrary,

$$Z(t) = (b-t)^{-\varkappa} e^{\Gamma(t)},$$

$$\Gamma(t) = \frac{1}{2\pi}\int_a^b \frac{\theta(\tau)\,d\tau}{\tau - t} = \frac{\operatorname{ctg}\mu\pi}{2}\theta(t) + \frac{(t-a)^{1-\mu}}{2\pi}\frac{d}{dt}\int_a^t \frac{d\tau}{(t-\tau)^{1-\mu}}\int_\tau^b \frac{\theta'(w)\,dw}{(w-\tau)^{\mu}(w-a)^{1-\mu}},$$

$$\theta(t) = \arg A(t), \qquad 0 < \theta(a) < 2\pi.$$

Similarly, for (8), when $K(x,t) \equiv 0$ and $\varkappa \leqslant 0$, a similar formula is obtained, likewise not containing integrals in the sense of principal value. When $K(x,t) \equiv 0$, the solvability conditions for equation (7) for $\varkappa < 0$ and for equation (8) for $\varkappa > 0$ are as follows:

$$\int_a^b \frac{f(x)x^{k-1}\,dx}{Z(x)(b-x)^{1-\mu}} = 0; \qquad \int_a^b v(t)(b-t)^{1-\mu} Z(t) t^j \left(\frac{d}{dt}\int_t^b \frac{g(x)\,dx}{(x-t)^{\mu}}\right) dt = 0$$

$$k = 1, \dots, -\varkappa, \qquad j = 1, \dots, \varkappa.$$

For the complete equation in the indicated spaces, the Noether theorems and the theorems on equivalent regularization are valid.

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Note: Figure translations are in progress. See original paper for figures.