A DISTANCE BETWEEN DISTRIBUTIONS CONNECTED WITH THEIR VALUES ON CONVEX SETS

MATHEMATICS

1967

SovietRxiv

Abstract

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UDC 519.214

MATHEMATICS

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A DISTANCE BETWEEN DISTRIBUTIONS CONNECTED WITH THEIR VALUES ON CONVEX SETS

(Presented by Academician A. N. Kolmogorov, December 17, 1966)

In the paper ([1]) the author obtained an inequality (a two-dimensional analogue of Esseen's inequality ([2]), § 39) that makes it possible to estimate, in terms of the difference of characteristic functions, the difference of the values $P(A) - Q(A)$ of the corresponding distributions for all rectangles A with sides parallel to the coordinate axes. The purpose of the present note is to derive inequality (4), which gives an upper estimate for the distance

$$\rho(P, Q) = \sup_A |P(A) - Q(A)|$$

(where the supremum is taken over all convex sets of the space R^s) in terms of characteristic functions. Let us note that a method for estimating the difference $P(A) - Q(A)$ for balls was indicated in ([3]) (see also ([4])).

The starting point in this and similar cases may be the following relation, easily derived from the inversion formula for Fourier integrals. Let ξ, η, and ζ be s-dimensional random vectors, ζ independent of ξ and η and having an absolutely integrable characteristic function $h(t)$. Then

$$\Delta_A = Pr\{\xi + \zeta \in A\} - Pr\{\eta + \zeta \in A\} = \frac{1}{(2\pi)^s} \int_{R^s} \overline{\tau_A(t)}(f(t) - g(t))h(t)\, dt.$$

Here A is an arbitrary bounded Borel set;

$$\tau_A(t) = \int_A e^{i(t,x)}\, dx;$$

f and g are the characteristic functions of the vectors ξ and η, respectively.

Representing the quantity $(2\pi)^s \Delta_A$ as the sum $I_1 + I_2 + I_3$, where

$$I_1 = \int_{|t| \le 1}, \qquad I_2 = \int_{1 < |t| < T}, \qquad I_3 = \int_{|t| \ge T},$$

and applying the Cauchy–Bunyakovsky inequality to each integral, we obtain

$$\begin{aligned}(2\pi)^s|\Delta_A| \le \left(\int_{|t|\le 1} |t|^2|\tau_A|^2\right)^{1/2} J_1 + \left(\int_{1<|t|<T} |\tau_A|^2\right) J_2 \\ + 2\left(\int_{|t|\ge T} |\tau_A|^2\right)^{1/2} J_3,\end{aligned} \qquad (1)$$

where

$$J_1 = \left(\int_{|t|\le 1} \frac{|f-g|^2}{|t|^2}\right)^{1/2}, \qquad J_2 = \left(\int_{1<|t|<T} |f-g|^2\right)^{1/2}, \qquad J_3 = \left(\int_{|t|\ge T} |h|^2\right)^{1/2}.$$

Let O_r denote the ball of radius r centered at the origin, and let

$$\Delta_r = \sup_{A\subset O_r} |\Delta_A|,$$

where the supremum is taken over all convex $A \subset O_r$. As is easy to see,

$$\gamma_1 = \sup_A |\Delta_A| \le 2\Delta_r + 2Pr\{\eta+\zeta \in \overline{O}_r\}.$$

The quantity Λ_r can be estimated with the aid of inequality (1). Indeed, if A is a convex set lying in the ball O_r, and $S(A)$ is the magnitude of its surface, then

$$S(A) \le S(O_r) \le 2\pi^{s/2}\Gamma^{-1}(s/2)r^{s-1}.$$

Therefore, taking into account the inequalities for the integrals of $\tau_A(t)$ (formulas of the paper ([1]), p. 379), we obtain, for $A \subset O_r$,

$$|\Delta_A| \le \frac{\sqrt{\lambda_s S(O_r)}}{(2\pi)^s}\left[J_1 + 2\sqrt{2}J_2 + \frac{4\sqrt{2}}{\sqrt{T}}J_3\right],$$

where

$$\lambda_s = \frac{(2\pi)^{s+1}}{4}\left(\int_0^\pi \sin^s\alpha\, d\alpha\right)^{-1}. \qquad (2)$$

Denote the right-hand side of (2) by Λ_r^*. Then

$$\gamma_1 \le 2\Lambda_r^* + 2Pr\{\eta+\zeta \in \overline{O}_r\}.$$

Here the constants r and T and the random variable ζ are at our disposal. The lemma stated below serves for passing from γ_1 to $\gamma = \rho(P, Q)$. Before formulating it, introduce the following notation: A is a convex set; $\dot{A}$ is its boundary; $(\dot{A})^\delta$ is the δ-neighborhood of A; $\omega_\eta(\delta) = \sup_A Pr\{\eta \in (\dot{A})^\delta\}$. Let us note that, for any A, the inequalities

$$Pr\{\xi \in A\} \leq Pr\{\xi + \zeta \in A \cup (\dot{A})^\delta\} + Pr\{|\zeta| \geq \delta\}, \tag{3}$$

$$Pr\{\xi \in A\} \geq Pr\{\xi + \zeta \in A \setminus (\dot{A})^\delta\} - Pr\{|\zeta| \geq \delta\}.$$

It is also easy to prove that $\omega_{\eta+\zeta}(\delta) \leq \omega_\eta(\delta)$.

Lemma. The inequalities hold

$$\gamma_1 \leq \gamma, \qquad \gamma \leq \gamma_1 + \omega_\eta(\delta) + 2Pr\{|\zeta| \geq \delta\}.$$

We omit the proof of the lemma.

Let us now note that

$$Pr\{\eta + \zeta \in \overline{O}_r\} \leq Pr\{(\eta + \zeta \in \overline{O}_r) \cap (|\zeta| < \delta)\} + Pr\{|\zeta| \geq \delta\} \leq$$

$$\leq Pr\{|\eta| \geq r\} + \omega_\eta(\delta) + Pr\{|\zeta| \geq \delta\}.$$

Therefore

$$\gamma = \rho(P, Q) \leq 2\Lambda_r^* + 2Pr\{|\eta| \geq r\} + 3\omega_\eta(\delta) + 4Pr\{|\zeta| \geq \delta\},$$

$$\rho(P, Q) \leq C_s r^{(s-1)/2} \left[J_1 + 2\sqrt{2} J_2 + \frac{4\sqrt{2}}{\sqrt{T}} J_3 \right] +$$

$$+3\omega_\eta(\delta) + 2Pr\{|\eta| \geq r\} + 4Pr\{|\zeta| \geq \delta\}, \tag{4}$$

where $C_s = \sqrt{\lambda_s S(O_1)}/(2\pi)^s$.

From Stirling' s formula it is not difficult to derive that

$$C_s = \sqrt[4]{2}\, [s(s+1)]^{1/4} \left(\frac{e}{2\pi s} \right)^{s/4} e^{\theta/s + \theta'/12s}, \qquad |\theta|,\ |\theta'| \leq 1.$$

The function $h(t)$ and the constants r, δ, and T entering inequality (4) may be chosen depending on our aims.

We shall now show how the function $\omega_\eta(\delta)$ can be estimated in the example of the normal distribution in R^s with density

$$p_\eta(x) = \frac{1}{(2\pi)^{s/2}} e^{-|x|^2/2}.$$

Let A be an arbitrary convex set in R^s. Without loss of generality, one may assume that A does not lie in any subspace of R^s of dimension $s' < s$. Consider the system of concentric spheres O_n, $n = 0, 1, 2, ...$, with centers at the origin and such that the radius of O_n is equal to n. The part of the surface $\dot{A}$ lying between the n-th and $(n+1)$-st spheres does not exceed its part lying inside the $(n+1)$-st sphere. Suppose that $\delta < 1$. Then the measure of the part of the δ-neighborhood of $\dot{A}$ lying between the n-th and $(n+1)$-st spheres does not exceed the measure of the δ-neighborhood of the $(n+1)$-st sphere, i.e., in any case, does not exceed $2\delta S(O_{n+2})$. The probability of hitting the mentioned part of the δ-neighborhood of $\dot{A}$ does not exceed

$$\frac{1}{(2\pi)^{s/2}} e^{-(n-1)^2/2} 2\delta \frac{2\pi^{s/2}}{\Gamma(s/2)} (n+2)^{s-1}$$

(for $n \geqslant 1$; for $n = 0$ the factor $e^{-(n-1)^2/2}$ must be replaced by 1). Therefore

$$Pr\{\eta \in (\dot{A})^\delta\} \leqslant \frac{4\delta}{2^{s/2}\Gamma(s/2)} \left[2^{s-1} + \sum_{n=1}^{\infty} e^{-(n-1)^2/2} (n+2)^{s-1} \right].$$

From this it is not difficult to derive that for $\delta < 1$

$$\omega_\eta(\delta) \leqslant \delta \frac{4}{2^{s/2}\Gamma(s/2)} \left[2^{s-1} + 3^{s-1} + 2^{3s/2-3}\Gamma(s/2) + 2^{3s-4} \frac{\sqrt{2\pi}}{2} \right].$$

From the last inequality and Stirling' s formula it follows that

$$\omega_\eta(\delta) \leqslant \delta C_{\text{abs}} \cdot 2^s,$$

where C_{abs} is an absolute constant.

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Received
13 XII 1966

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Note: Figure translations are in progress. See original paper for figures.