Systems of differential equations whose critical singular points are fixed

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Abstract

Full Text

Preamble

DIFFERENTIAL EQUATIONS 1967, Vol. III, No. 3 SYSTEMS OF DIFFERENTIAL EQUATIONS WITH FIXED CRITICAL SINGULAR POINTS

Consider a system of differential equations given by:

$$\frac{dx}{dz} = P_1(x, y, z), \quad \frac{dy}{dz} = P_2(x, y, z)$$

where $Q_i(x, y, z)$ are polynomials in the variables x and y with coefficients that are analytic with respect to z and holomorphic in a certain domain. The expressions $\frac{P_1}{Q_1}$ and $\frac{P_2}{Q_2}$ are irreducible rational functions. The objective is to determine the conditions that the system must satisfy such that both components of the solution, $x(z)$ and $y(z)$, do not contain movable critical singular points. Let us first assume and represent the denominators respectively in the form:

$$Q_i(y, z) = \prod [y - \alpha_k(z)]^{m_k}$$

Y1 + Y2 + - + Ys = Af > 1 , Y1Y2 • • • Y s ¥ = 0 ;

$\sigma_1 + \sigma_2 + \cdots + \sigma_r = m > l, \quad \sigma_1 \ldots \sigma_r = \mu \neq 0;$

$P_1(y, z) = \sum P_k(z) y^k, \quad P_2(x, z) = \sum f_k(z) x^k$

where these functions are holomorphic in some domain $(z) \in D$. Taking the functions $y - a(z)$ and $x - b(z)$ as the new unknowns, respectively, and retaining the original notation, we can write the system in the form

/=2 /=2

In the following sections, we shall restrict ourselves to system transformations such that the existence of movable critical singular points in the transformed system necessarily implies their existence in the original system.

Suppose that $P \neq 0$,

$y_i > 0, \quad \chi > 1.$ (3)

We introduce a parameter into system (1) (see p. 160):

$x = a^{\chi-1}\tau, \quad y = a^{\chi-1}w, \quad z = z_0 + \lambda y^{\chi-1}u,$

$\frac{dz}{d\tau} = 0$ $(i = 2, k = 2, ...)$. In the resulting system, we set $\lambda = 0$; then

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For the case where $l = 2$ $(k = 0)$, we have $a \neq 0$ and $b \neq 0$. This system possesses a one-parameter family of solutions:

$$x = L(t + C)^{k_1}, \quad y = K(t + C)^{k_2}$$

where L and K are constant numbers, and C is an arbitrary parameter. It is evident that if at least one exponent k_i in (5) is not an integer, then (5) contains critical movable singular points. Consequently (as noted in the footnote), the system (2) also possesses movable critical singularities. However, from (3) it follows that···

0 , < 0 .

Thus, equation (3) ensures the existence of critical moving singular points for system (2). Now, let us assume that

$Y_i =$ ... and $b_1 > i$: (6)

We write system (2) in the form

J k =0 k =0

where $Q \neq 0$, and all functions are holomorphic in some domain. We introduce a parameter λ into system (7). Remark: In all subsequent discussions, we shall take values from a certain sub-domain of the domain in which the functions possess the required properties. Differential equations

we obtain the solution to this system, which we seek in the form. It is easy to see that $s = 0, 1, 2, ...$ To determine these values, we have a system. From the first equation of this system,

$$s_0(z_0) = C_1 \ln(A(f_0) \ln(z + C_2)).$$

It follows that condition (6) also ensures the presence of moving critical singular points in system (2). We formulate the obtained results as follows: Lemma 1. If $\frac{\partial Q}{\partial z} \neq 0$, $\frac{\partial P}{\partial z} \neq 0$, $Q \neq 0$, and $P \neq 0$, then for system (2) to have only single-valued moving singular points, it is necessary that $\gamma_1 = \cdots = \gamma_n = \delta_1 = \cdots = \delta_n = 0$. Suppose the conditions of Lemma 1 are satisfied; then system (2) can be written in the form $\frac{dy}{dz} = ...$ We introduce a parameter into (8)

$$x = \tau, \quad y = \lambda w, \quad z = z_0 + \lambda t$$

and set $\lambda = 0$ in the resulting system; then

$dT = r_J dW_0 P_2(T_0, z_0)$

By eliminating from this system, we obtain:

In order for the system to be free of movable critical singular points, it is necessary ([1], p. 166) that Q decomposes with respect to no more than four factors, and the degree must be lower. Consequently, the decomposition of $z) = ...$ into partial fractions must contain no polynomial part.

$P_2(x, z_0)$

Remark. In the lemma, it was assumed that $z \neq 0$. This property is utilized in the reasoning regarding (9); therefore, even when $z = 0$, the absence of a polynomial part in the expansion is necessary. In this case, the result depends on the following:

Lemma. For the system to lack movable critical singular points, it is necessary that the system can be represented in one of the following forms: 1. $y - a(z)\phi a_i(z)$, where $(z) \neq \beta_i(z)$ and $(z) \neq 0$; 2. $(z)y^1$, where any rational function contains no linear factors; 3. $y = P(z)$, where $P(z)$ is an arbitrary rational function.

Suppose we have a system of the form (10); we introduce a parameter into it.

x–fa(z) = KT, y-aj(z)=lW,

For $K = 0$, we obtain:

The one-parameter family of solutions for this system is given by $Z(t) = K(t+C)$ and $Z(t) + B = L(t + C)$. In system (10), the roles can be interchanged, as was done in the formulation of the lemma relative to the proof.

$T_0 = \exp K_1(t + C)$, $H_0 = \exp L_1(t + C)$,

if $\frac{K}{L} = \frac{K_1}{L_1}$. Here K, K_1, L, L_1 are non-zero constants, and C is an arbitrary parameter. From this, it is evident that to ensure the absence of moving critical singularities, it is necessary that n_j are integers and $m_j = 1 - n_j$. We now introduce the parameter λ into the system as follows:

$x = T, \quad y - a_1(z) = \lambda W, \quad z = z_0 + U.$

For $\lambda = 0$, we obtain:

Eliminating the variables, we obtain the following. It is evident that, as can be easily seen, we then have $s > ....$ Suppose that $(M-1)(m-1) \geq 0$, and there exists at least one pair for which the value is -1. It is clear that one can always set the values such that $N_k(z)$ can be written as:

$k = 1 \dots n_k, \quad k = 1, 2, \dots, m$

$A_k = \pm \frac{t-1}{A - A_k}, \quad k = 2, \dots, M$

Furthermore, some values of m may be equal to ∞ (this corresponds to cases where the variables are unbounded). Taking this into account, the system can be transformed by a change of the independent variable into the following form:

$y - a$. Taking into account (16), equation (14) can be written as:

T O –Pfe K = R + L

Now, let us introduce the parameter λ into equation (16):

$$x' = \lambda x, \quad y' = y, \quad z' = z_0 + kt$$

By setting $\lambda = 0$ and reducing the system to a single second-order differential equation with respect to x, we obtain $\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = f(t)$. For equations (17) and (18), we respectively derive the first integrals:

$$n = 1, \quad n = r + 1$$

{v,-a fa>-a*).

£ = 2 &=s+l

By virtue of the problem being solved, equations (19) and (20) are Briot-Bouquet equations ([1], p. 114); consequently, the exponents of the binomials can only take on specific sets of values. If < 0, then in (20), and if < 0, then in (19), without loss of generality, one may assume that system (16) can be written in the form:

where $h_x > 2$, and it is easily seen [1] that $h < 3$.

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If we assume $r < m$ and rewrite (19) as

then we obtain either 2 or $\cdots > 2m$. The first case is excluded, while the second indicates the presence of critical moving singularities. Consequently, only $r = m$ is possible. By performing the substitution $z = ...$ in system (21) and constructing formulas analogous to (19) and (20) for any new system, we conclude that either $= 1$ or one of the equations (19), (20) cannot be a Briot-Bouquet equation. Thus, we have proven Lemma 3: if system (10) is such that $(M - 1) \neq 0$ and there exist such $A_j(z)$, then this system contains moving critical singular points. Assuming $- ... A = \cdots = -B$ in (19) and (20), we obtain the corresponding results. It is evident that under the identified conditions, the system admits a first integral:

$$(z - \alpha)(z_1 - \alpha)(z - \beta)(z_1 - \beta) = -1$$

After reducing (10) (taking the above into account) to a single equation using (22), we have:

$$+4C$$

$$\frac{dz}{dJ} = C(K - \alpha^2)$$

Equation (23) can be integrated using elliptic functions; consequently, it is a single-valued function. From this, it is easily seen that the variable y is also a single-valued function. This leads to the following theorem:

Theorem: Among the systems that can be reduced to the form L, there is only one system that possesses no critical moving singular points.

Finally, let us consider the case for system (10) where $l(m - 1) = 0$. Without loss of generality, we may assume:

1. Then the system (10) can be

reduced to the form

where some values may be equal. System (25) possesses a first integral $(1 - \mu)\, y - a$. By reducing the order of system (25), we obtain the equation

$= P(x)$ (where P is a polynomial),

which must belong to the class of Briot-Bouquet equations. The possible values of m and their corresponding λ_i will be determined. Specifically: I. For $m = 1$, λ can take the values: -1 or any integer. II. For $m = 2$, the pairs (λ_1, λ_2) can take the values: $(n, -n)$ where n is an integer; (∞, ∞); $(2, 2)$; $(2, 2, 3)$; $(2, 6)$; $(3, 6)$; $(2, 4)$; $(4, 4)$; $(3, 3)$. III. For $m = 3$, the triplets $(\lambda_1, \lambda_2, \lambda_3)$ can take the values: $(2, 3, 6)$; $(2, 4, 4)$; $(3, 3, 3)$; $(2, 2, 2)$. IV. For $m = 4 ...$ It is evident from (25) that the absence of movable multi-valued singular points in the components implies their absence in z. In this case, x and y are expressed in terms of elementary and elliptic functions.

Theorem 2. For system (10) to have no movable critical singular points, it is necessary and sufficient that, through a transformation of the independent variable and a first integral of the form (26), the system reduces to a Briot-Bouquet equation; specifically, that m and the corresponding λ_i in (25) take the specified values.

Let us consider a system of the form (11), and assume first that $\lambda \neq 0$. Then $z_i = \psi_i(x, z_j)$ for $i \neq j$. It can always be assumed that

Introduction

Into the system, the parameters $y = W$ and $z = z$ are introduced, and at···

0 we obtain

/(2o)f / = 2

a ^ 0 , p k (z 0)=£0.

By utilizing the first integral

$1 - \beta \sum p_k(z) W_{k+l} = C$

we arrive at the equation $(-\delta_i + \mathcal{J}_k) W_0$

L k =0

Jablonskii established that for the absence of movable critical singular points, it is necessary that $n = 2$; however, this condition may not be sufficient. L e m m a: For system (11) to be free of movable critical singular points, it is necessary that $n = 2$. Suppose the lemma holds; that is, $n = 2$. If $n = N = 1$, the system is linear, which implies it contains no movable singular points at all.

Suppose $nN \neq 1$. We introduce the parameter λ:

- —-,n +i ~ • # ~ , J V + i ™ > 2 –' z o + ^

For $X = 0$, we obtain:

Consider a one-parameter family of solutions for the system $A(t + C)B(t + \dots)$, where C is a constant and an arbitrary parameter. To ensure uniqueness, it is necessary to require that $N, n-1, s$, and l be integers. It is easily seen that the latter is possible only for the following values:

$n = 2, n = 3, N = 3$. Suppose we have $n = 2$ and $s = 1$; that is:

-f' Po ~ - =

P 2 (z) # 0 , 4l (x)=£0.

This system can be reduced via the linear transformation $z = y + p(x)y$ to the form $z' = x$, which is clearly equivalent to a second-order equation.

J^=j 2 (z)y* + b(z)y. (33) dz 2

Theorem 3. Equation (33), and consequently the system, contains movable critical singular points if and only if the coefficients are constant, or if by the transformation $y = \lambda(z)W + \phi(z)$, where

$$\lambda(z) = C\rho^2(z), \quad \phi(z) = \int \rho^2(z)dz,$$

$v = -\lambda(z) - b\rho^2(z) \int \rho(z)dz$, equation (33) is reduced to

$$\frac{dW}{dz} = 6W^2 + az + b,$$

where a, b, and C are constant numbers. In this case, the solution of the system is expressed through elementary functions, elliptic functions, or the Painlevé transcendents (specifically, solutions to the first Painlevé equation). **Theorem 3' .** In order for equation (33), and therefore system (32), to be free of movable critical singular points when $\rho_1 \neq$ const and $\rho_2 \neq$ const, it is necessary and sufficient that $f(z) = az + b$, where a and b are certain constant numbers. Theorem 3' follows from Theorem 3 and the form of equation (33) after applying transformation (34).

Suppose we have $n = 3, N = 1$, i.e.,

The equation $\mathrm{P}_3(z)y''' + \mathrm{P}_2(z)y'' + \mathrm{P}_1(z)y' = q_i(z)x$ (35) can be reduced to the form $y''' = \mathrm{P}_3(z)y'' + \mathrm{P}_2(z)y' + \mathrm{P}_1(z)y$, which is equivalent to the equation $\mathrm{P}_2(z)y'' + \mathrm{P}_1(z)y'$. Analogous to the case where $n = 2$ and $m = 1$, we have the following:

Theorem 4. Equation (37), and consequently system (36), do not contain movable critical singular points if and only if the coefficients are constant. This can be achieved via the transformation $y = y_1(z)W + v(z)$. If $\delta \neq 0$, then by choosing the appropriate parameters, one can ensure $\delta = 0$, and by choosing $\alpha = 1$.

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$T(z)dz$, where C and a are constant values. In this case, the solutions of the system are expressed through elementary functions, elliptic functions, and Painlevé transcendents (specifically, solutions to the second Painlevé equation). Theorem 4: For equation (37), and consequently system (36), to be free of movable critical singular points $(const, const, const)$, it is necessary and sufficient that $P_3(z) - 12P_1(z)P_2(z) - \cdots + 6P_1(z)P_2(z) - \ldots$

$-8/3 - 36P_3(z)\left[aC_1 \int P_3(z)dz + bC_2\right] = 0;$

$9P_1(z)P(z)P_1'(z) + 2P_1^3(z)P(z) - P(z)P_1(z)$

2 (z) + p 2 (z) p7 (z) - 27a C 3 p f / 6 (z) = 0,

$p_3(z)p$, where a, b, c are constant numbers. This case was previously examined in [?]. Suppose system (1) is written in the form (12); it can then be reduced to $Q(x, z)$. The general solution of this system is:

$$x = C \int dz + C$$

If $Q = \Pi - P/W_1$ in the case where $\frac{dQ}{dz} \neq 0$, and among the values of δ there are equal values for which $p_y(z) = \text{const}$, then the solution $y = y(z, C)$ contains moving critical singular points of a logarithmic nature.

Theorem 5

For system (39) to be free of moving critical singular points, it is necessary and sufficient that one of the following conditions be satisfied: $\frac{dQ}{dz} = 0$. If $\frac{dQ}{dz} \neq 0$, then for the constant $\delta_i > 2$.

The validity of this theorem can be verified by direct calculation of the quadrature in (40). Ultimately, Theorems 1, 2, 3, 4, 5, and the theorem presented in [?] provide the necessary and sufficient conditions for the absence of critical moving singular points in nonlinear systems of the form (1). Consequently, it is possible to conclude···

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Figures

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THE CONCEPT OF STABILITY WITH RESPECT TO GENERALIZED PERTURBATIONS

S. T. ZAVALISHCHIN

Introduction. The emergence of perturbations, violating the desired character of the development of a real process, always proceeds in time. However, in determining the corresponding deviations of the process, the time of emergence of impacts is usually neglected [6]. As a result, when analyzing automatic control systems, idealizations of the Heaviside function type, impulsive and other functions appear. Therefore, one has has to encounter the following problems:

1) finding the reactions of the object to perturbations containing derivatives of some orders from discontinuous functions;

2) clarification of conditions under which small input signals, having as components derivatives of some orders from discontinuous functions, are transformed into small output signals.

Connected with the latter question is the no less difficult problem of determining the concept of "smallness" in application to the processes under discussion. An attempt to solve all these questions by means of classical analysis is devoid of internal logic and often [7] leads to incorrect results. A way out of this contradiction is to involve the apparatus of generalized functions, which allows obtaining information consistent with experimental data. On this path, in monographs [6—8], possible methods for determining the reaction of a system to systemo generalized perturbations belonging to certain special classes are alnore discussed. In book [8] and article [9], questions of the asymptotic behavariior of the system's reaction to a series of impulse perturbations are partially touched upon. However, the first fundamental research in this direction was carried out by E. A. Barbashin [1]. In the class of impacts, which are generalized derivatives of functions of locally bounded variatimvan, subclasses were found that are capable of being parried by a system possessing one or another type of its own stability.

In this same article, the problem of stability with respect to perturbations of a more complex nature — generalized processes — is posed. The main difficulty that arises in solving this question is connected with the problem of estimating the magnitude of a generalized function. The reason for such a complication is that, unlike classical processes, the value of a generalized process at individual moments in time, generally speaking, cannot be given any clear quantitative meaning. Characterizing the situation as a whole, it is necessary to note that the more complex the structure of the attracted generalized functions, the more abstract the concept of "small" generalized process becomes. Thus, for example, in article [4], where arbitrary generalized functions were taken as permissible impacts, it was possible to introduce only the concept of an infinitely small process as a sequence of generalized ones, disappearing according to some rule.

Figure 1: Figure 1

functions. In this, the study of sctobilitcity the systems was reduced to the clarifying the contnuitnocty mappearing the genolmenero ppoctpanctne, it specifies processes, dadawed with this or that tipe of condagence. On this path, in note [4], the constue of sctoblicuity with throcter to genolmenen effectes was inredened. Honever, devoting touat most of the atimation to the matematavical stope of the ragany, the author omitted the presentation of the classicallely consited sctoibilooctns, that served as the starting poant in oppedelining the types of sctobillies treappasizations arrieed on genolmeness pertubation, carried out by dynamicaneous systemes. The purpose of this note is to fill the resultaning gap, t.e., to ylorrify the classivecker codepwate the constue of sctoibiltocity with thoceter to genolmenesx pertubation, inredefined in [4].

§ 1. DISCUSSION OF CLASSICALLY STABLE SYSTEMS FROM THE POINT OF VIEW OF THE TOPOLOGICAL CHARACTER OF THE MAPPINGS OF THE FUNCTION SPOCTPANICA ΦYNKЦUN LOCALLY OF FINITH VARIATION SPECIFIED BY THEM *)

Let as giom a system of differential equations be given

$$\mu = A(t)\,\mu, \tag{1.1}$$

where μ is an n-dimension. vector; $A(t)$ is an $n \times n$ matrix with variable elements. We will asssume that the matrinx $A(t)$ is takeh, that the solution sadaw Cauchy is possible in the eclire Euclidon prostpancters E_n and is continuable for $t \geq 0$. Torn the solution with the initial costonnment μ_0 man be repressented in fum $\mu = U(t)\mu_0$, yhere vhere $U(t)$ denoates the fundamental solution matrucy peuuetion of sustem (1.1). Along with system (1.1), we will new pacsmder the perturbed systemy, the equations of whorich we write in the form of the collylowing differential relation:

$$d\mu = A(t)\,dt\,\mu + d\eta. \tag{1.2}$$

In system (1.2), the function η describes the perturbing ssfect. Let us assssume, that it bealongs to the prostpancte $V(\infty)$. Torgas solution to the sadawe Cauchy existen and is given by the formys of the Cauchy, which, unusing the constitt of the integral Stieltjes, we wite in the bum

$$\mu(t) = W(t, 0)\,\mu_0 + \int_0^t W(t, s)\,d\eta(s). \tag{1.3}$$

In relation (1.3), $W(t, s) = U(t)\,U^{-1}(s)$ — operator Cauchy.

In the sadave yesrobility w retosocetorn to costoannly deectrymening perturbation, pasemitends are beicthes, that have goctatowly gooued didifferentialines conortanories (hanpumly, absoluetly diffferentianpysiable), it is asssumes, tho the navalled moment spement sustema hs reducen at navale koordinant, t. e. $\mu_0 = 0$.

Let us turn to stability according to Lyananov [3]. In this clyvae we have $\eta(t) \equiv 0$ for $0 < t < \infty$ and in formyle (1.3) the integraled vrem ucveetert. Odnaver, the first vrem ansovers the expression [1]

$$W(t, 0)\,\mu_0 = \int_0^t W(t, s)\,d\chi(s)\,\mu_0, \quad t > 0,$$

*) As in papers [4, 5], in this note we will deonate such a spoctemontare space by $V(\infty)$; at the same time, if $\eta \in V(\infty)$, then $\eta(t) = 0$ for $t \leq 0$.

Figure 2: Figure 2

where $\chi(s)$ is the Heaviside function (unit step). Hence, the stability of the zero solution of system (1.1) in the Lyapunov sense can be treated as stability with respect to perturbations of the Heaviside function type. Thus, the problem of Lyapunov stability is immersed [2] into the problem of stability under constantly acting perturbations, if the permissible class of the latter is extended to the space $V(\infty)$. Assuming that η is an arbitrary element of this space, for the reaction of system (1.2) to this influence, we will use the expression

$$\mu = \int_0^t W(t, s)\,\mathrm{d}\eta(s). \tag{1.4}$$

In work [1], relation (1.4) is treated as an operator mapping some functional space D of elements from $V(\infty)$ into the space D_0 of functions from $V(\infty)$. At the same time, certain types of asymptotic behavior of trajectories of the free system (1.1) are characterized in terms of spaces D_0, D. These results serve as the starting material in defining such an approach to the concept of stability, which allows for formulating the concept of stability of generalized processes. Namely, we will consider the Cauchy formula (1.4) as a transformation of the space $V(\infty)$ into itself. Then with each of the types of spaces D_0 and D considered in [1], we will associate some special topologiy in in the space $V(\infty)$. This correspondence can be arranged in such a way tha that the requirement that the operator (1.4) map one or another space D into the space D_0, D, turns out to be equivalent to the requirement of continuity of the operator (1.4) when mapping the space $V(\infty)$ (endowed with the topology corresponding to the type of space D) into the space $V(\infty)$ with the topology corresponding to the space D_0. Thereby, the above-mentioned types of asymptotic behavior of the trajectory of system (1.1) will be characterized in terms of types of continuity of the transformation (1.4) of the space $V(\infty)$ into itself. On this path, the concept of stability is freed from the now irrelevant concept of the state of the process and appears as a natural starting point for a view on stability in an attempt to define the concept of stability of generalized processes.

Now, in accordance with the sketched plan, we proceed to the discussion of the main results [1] concerning the dependencies between the classical types of stability of system (1.1) and types of functional spaces D_0, D, and D. Preliminarily, it is necessary to define a number of subspaces of the space $V(\infty)$. The set of functions from $V(\infty)$, for which

$$\|\eta\|_B = \sup_{t\to 0} \|\eta(t)\|_{E_n} < \infty, \tag{1.5}$$

we will denote by B. The space of functions having bounded variation on the half-line $[0, \infty)$ will be denoted by $V(0, \infty)$. Those functions from the space $V(\infty)$, for which

$$\|\eta\|_V = \sup_{t\to 0} \bigvee_t^{t+1} \eta < \infty \quad \left(\bigvee_\alpha^B \eta \text{ — variation of } \eta\right), \tag{1.6}$$

constitute the space ν [1]. Obviously, the inclusions $V(0, \infty) \subset \nu$, $V(0, \infty) \subset B$ hold. The spaces B and ν intersect. For example, the function $\eta = t$ belongs to the space ν, but $\eta \in B$. On the other hand, the function

Figure 3: Figure 3

duits D_0-convergence. Postomy uniajeter no determines to D-convergence such k_0, that $\Omega\eta\ k_0 \in D_0$. But $\Omega\eta_m = \frac{1}{k_0}\Omega\eta$ in, takem obpasan, Oη takine припадлment рространству D_D, что протоверечег предположенне. Octaetcs установить, что onepator Ω является ограннченным. Так как onepator Q отображает пространство D_l и пространство D_0, то нз того, что of D_l-непрермееп, вытеляет eco непрерывность прн отображенни в пространство D_0 пространство D_l. Теперь остается проночнть известную теорему функционального аналнза о том, что лннейный оператор, непрерыано отображающий одно нормнрованное пространство в другое, является ограпнченпым.

Достаточность. Нусть посгедовательность $\eta_1, \eta_2, \ldots (\eta_k \in C_l)$ D_l-еходител. Тогда о некоторого потера k_0 $\eta_k \in D_l$ it $\|\eta_\lambda\|_{O_l} \to 0$ прн $k \to \infty$ $(k > k_0)$. На того, что $\Omega D_l \in D_0$, вытекет, что посгедовательность $(\Omega\eta_k)$, начннан с потера k_0 прннадлежнт rгостраиству D_0. Так как оператор Ω огранпчен, то $\|\Omega\eta_k\|_{O_0} \leqslant \|\Omega\|+\|\eta_k\|_{O_l}$, прн $k > k_0$. Otcioda вытекает, что $\|\Omega\eta_k\|_{O_0} \to 0$ прн $k \to \infty$, $k \geqslant k_0$. Последнее завершает доказательство теоремы.

Возяратнмся к нашей задаче. Введем в пространство $V(\infty)$ pad понннтий сходнмостн.

Определенне 1. Посгедовательносts $\eta_1, \eta_2, \ldots (\eta_k \in V(\infty))$ nasonon B-есодишеlicя, если прн $k \to$ со pahnomepo по a $(a = 1,2, \ldots)$ стремнтся к hynlo $\sup\limits_{0<t<a} \|\eta_0(0\|_{c_a}$; та же пестедовательносtь будет называтьсй V-ссодащейся, если прн $k \to \infty$ pahnomepo to a стрешнтся it hynlo $\overset{a}{\underset{0}{V}}\eta_k$; наконец, скажем, что пестедовательность $\eta_1, \eta_2, \ldots$ ν-сходитск, если прн $k \to \infty$ pahnomepo по a стремнтся it hynlo $\max\limits_{0<l<a} \overset{l+1}{\underset{l}{V}}\eta_k$.

Выяснм соотоощенне введенных сходимостей. Очевндно, V-сходимость спльнее ν-сходнмости и B-сходимостн. Делее, пространство $V(\infty)$, являесь сопряженным н пространству $c(0, \infty)$ пепрерывпых фииитных функций со значеннями к пространстве E_1, имеет топологню слабой сходнмостн. Сравннкак, ν-сходнмость н V-cxoditnocть со слабой сходимостью, можно установнть, что последная слабее первых. Для этого в сняу следанного выше замачання достаточно убедиться в справедлнвости этого факта для ν-сходимости. Привлекая ннтегральное представленне лнпейных непрерывпых функцноналов на пространстве $c(0, \infty)$ с учетом перавенства для ннтегралов Стилтьеса, будем нметь

$$\|\eta_6(x)\|_{\substack{e_n \\ x\in c(0,\infty)}} = \| \int_0^{l(z)} x(t)\, d\eta_a(t)\|_{e_n} \leqslant \max_{0<t<l(s)} |x(t)| \cdot \overset{l(c)}{\underset{0}{V}}\eta_k \leqslant$$

$$\leqslant (1 + |l(x)|)\|x\|_{k(0,\infty)} \cdot \max_{0<l<[l(x)]} \overset{l+1}{\underset{l}{V}}\eta_k.$$

В этих quenkax xepes $t(x)$ обозначенна konstanta, kotopaя удовлетвераot условня: $x(t) = 0$, ectil $t \geqslant t(x)$. То предположенню $\max\limits_{0<l<[l(x)]} \overset{l+1}{\underset{l}{V}}\eta_k$ стремнтся х пулю при $k \to \infty$. Hostomy стренится х пулю it $\eta_a(t)$, что it надлежало установнть.

В дальнейшем пространство $V(\infty)$, hadenennoe B-cxodimoctao, V-cxodimoctao, B-cxodimoctao, будем соответствено обозначать $BV(\infty)$, $VV(\infty)$, $oV(\infty)$.

Относительно асумптотнческнх свойстs сходящнхся последовательноcted in только the указанных пространствах справедлива

Figure 4: Figure 4

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Figure 5: Figure 5

$$\left.\begin{aligned} \frac{dx}{dz} &= \frac{1}{y-\alpha_1} + \\ &+ \sum_{k=2}^{s} \frac{1-m_k}{m_k}\,\frac{1-n_1}{n_1}\,\frac{1}{y-\alpha_k} - \frac{1-n_1}{n_1}\sum_{k=s+1}^{M}\frac{1}{y-\alpha_k} \\ \frac{dy}{dz} &= \sum_{k=1}^{s}\frac{1-n_k}{n_k}\,\frac{1}{x-\beta_k} - \sum_{k=r+1}^{m}\frac{1}{x-\beta_k}\,. \end{aligned}\right\} \tag{16}$$

Considering (16), (14) is written as

$$\frac{d^T T_0}{dt^2} = -\left(\frac{dT_0}{dt^2}\right)^2 \left[\sum_{k=1}^{r}\frac{1-n_k}{n_k}\,\frac{1}{T_0-\beta_k} - \sum_{k=r+1}^{m}\frac{1}{T_0-\beta_k}\right]. \tag{17}$$

Now let's introduce the parameter λ into (16):

$$x-\beta_1=\lambda u,\quad y=v,\quad z=z_0+\lambda t,$$

let $\lambda=0$ and transition to a single second-orot differential equation with poro regard with respect to v_0, we obtain

$$\frac{d^v v_0}{dt^2} = -\left(\frac{dv_b}{dt}\right)^2 \times$$

$$\times\left[\frac{n_1}{1-n_1}\,\frac{1}{v_0-\alpha_1} + \sum_{k=2}^{s}\frac{1-m_k}{m_k}\,\frac{1}{v_k-\alpha_k} - \sum_{k=r+1}^{M}\frac{1}{v_0-\alpha_k}\right]. \tag{18}$$

For (17) and (18) reaspectively we have the first integrals

$$\frac{dT_0}{dt} = C\prod_{k=1}^{r}(T_0-\beta_k)^{t-\frac{1}{n_k}}\prod_{k=r+1}^{m}(T_0-\beta_k), \tag{19}$$

$$\frac{dv_0}{dt} = C\,(v_0-\alpha_1)^{t-\frac{1}{1-n_1}}\prod_{k=1}^{s}(v_0-\alpha_k)^{t-\frac{1}{m_k}}\prod_{k=s+1}^{M}(v_0-\alpha_k). \tag{20}$$

In view of the problem being solved (19) and (20) ([1], p. 114) — the equations of Briot and Bouquet, it means, the exponents of the binomials $(T_0-\alpha_k)$ and $(v_0-\beta_k)$ can take completely definite groups of values. If $n_1>1$, then $1-n_1<0$ and, consequently, in (20) $s=M=2$, if, however, $n_1<0$, then in (19) $r=m=2$.

Without loss of generality, it can be assumed that $s=M=2$, then the system (16) is written in the form

$$\begin{aligned} \frac{dx}{dz} &= \frac{1}{y-\alpha_1} + \frac{n_1-2}{n_1}\,\frac{1}{y-\alpha_2}\,, \\ \frac{dy}{dz} &= \sum_{k=1}^{r}\frac{1-n_k}{n_k}\,\frac{1}{x-\beta_k} - \sum_{k=r+1}^{m}\frac{1}{x-\beta_k}\,, \end{aligned} \tag{21}$$

where $n_1>2$ and it is easy to see [1], that $m\leqslant 3$.

Figure 6: Figure 6

If we assume $r < m$ and in (19) write

$$\left(\frac{dT_0}{dt}\right)^N = \prod_{k=1}^{M} (T_0 - \beta_k)^{N_k},$$

then we obtain either $n_1 = 2$, or $\sum_{k=1}^{m} N_k > 2m$, but the first is excluded, and the second speaks to the presence of critical mobile singularities in $T_0(t)$. Therefore, it is possible only that $r = m$. Having made the substitus $z = \frac{n_1}{(1-n_1)} z'$ in system (21) and for any new system having constructed formulas, (19) andz, we come to the fact that either $n_1 = 1$, or one of equations (19), (20) carot be Brio and Bouquet. Thus is proven

Lemma 3. If the system (10) is such that $(M-1)(m-1) \neq 0$ and there exist such $A_i(z)$ and $B_i(z)$, that $A_i/B_i \neq -1$, then this system contains mobile critical singular points.

If we assume $A_1 = A_2 = \ldots = A_m = -B_1 = \ldots = -B_M$, then from (19) and (20), putting $n_k = \infty$, $m_k = \infty$, we obtain respectively

$$\frac{dT_0}{dt} = C \prod_{k=1}^{M} (T_0 - \alpha_k), \qquad (19_1)$$

$$\frac{dv_0}{dt} = C \prod_{k=1}^{m} (T_0 - \beta_k). \qquad (20_1)$$

Obviously $m = M = 2$, under the found conditions the system (10) admits the first integral

$$(y - \alpha_1)(y - \alpha_2)(x - \beta_1)(x - \beta_2) = \frac{1}{C} . \qquad (22)$$

After reduction of (10) (taking into account the aforesaid) with the help of (22) to a single equation we have

$$\left(\frac{dx}{dz}\right)^2 = C^2(\alpha_1 - \alpha_2)^2 (x - \beta_1)^2 (x - \beta_2)^2 + 4C(x - \beta_1)(x - \beta_2). \qquad (23)$$

Equation (23) integrates in elliptic functions, consequently, $x(z)$ is a single-valued function. But then it is easy to see that $y(z)$ is also a single-valued function.

Proven

Theorem 1. From systems of the form (10) for $(M-1)(m-1) \neq 0$ one and only one system, which can be reduced to the form

$$\frac{dx}{dz} = \frac{1}{y - \alpha_1} + \frac{1}{y - \alpha_2}, \qquad \frac{dy}{dz} = \frac{1}{x - \beta_1} + \frac{1}{x - \beta_2}, \qquad (24)$$

does not have critical mobile singular points.

Let, finally, in system (10) $(M-1)(m-1) = 0$. Without reducing generality, we can set $M = 1$. Then system (10) can be reduced to the form

$$\frac{dx}{dz} = \frac{1}{y - \alpha}, \qquad \frac{dy}{dz} = \sum_{k=1}^{m} \frac{1 - n_k}{n_k} \frac{1}{x - \beta_k}, \qquad (25)$$

Figure 7: Figure 7

where some n_k can be equal to ∞ as well. System (25) has a first integral

$$y - a = C \prod_{k=1}^{m} (x - \beta_k)^{-\left(1 - \frac{1}{n_k}\right)}. \quad (26)$$

By lowering the order of system (25), we obtain the equation

$$\left(\frac{dx}{dz}\right)^N = P(x) \quad (P \text{ — polynomial}),$$

which must belong to the class of Briot and Bouquet equations. This will find the possible values of m and the corresponding n_k. Namely,

I for $m = 1$ n_1 can take the values: -1, n — any integer, ∞;

II for $m = 2$ n_1 and n_2 can take respectively: n, $-n$ — integer;

II for $m = 3$ n_1 and n_2 can take respectively: n, $-n$ — integer; ∞, 2, ∞, ∞; 2, 2; 2, ∞; 2, 3; 2, 6; 3, 6; 2, 4; 4, 4; 3, 3;

III for $m = 3$ n_1, n_2 and n_3 can take respectively the values: 2, 2, ∞; 2, 3, 6; 2, 4, 4; 3, 3, 3; 2, 2, 2;

IV for $m = 4$ $n_1 = n_2 = n_3 = n_4 = 2$.

From (25) it is obvious that the absence of movable multivalued singular points for the component $x(z)$ entails the absence of such for $y(z)$ as well. In this case, x and y will be expressed in elementary and elliptic functions.

Theorem 2. *For the absence of movable critical singular points in system (10) with $(M-1)(m-1) = 0$, it is necessary and sufficient that by means of a transformation of the independent variable and a first integral of the form (26) the system is reduced to a Briot and Bouquet equation, namely, that in (25) m and the corresponding n_k take the values given above.*

Consider a system of the form (11), and let first $\dfrac{\partial Q_2}{\partial x} \not\equiv 0$. Then

$$Q_2(x, z) = \prod_{j=1}^{s} [x - \beta_j(z)]^{\delta_j}, \quad \delta_j \geqslant 2, \quad \beta_i(z) \not\equiv \beta_j(z), \quad i \neq j.$$

It is always possible to assume $\beta_1(z) \equiv 0$. Let us introduce a parameter λ into the system: $x = \lambda^{\delta_1} T$, $y = W$, $z = z_0 + \lambda^{\delta_1} t$ and for $\lambda = 0$ we obtain

$$\frac{dT_0}{dt} = \sum_{k=0}^{n} p_k(z_0) W_0^k, \quad \frac{dW_0}{dt} = \frac{P_2(0, z_0)}{T_0^{\delta_1} \prod_{j=2}^{s} [-\beta_j(z_0)]^{\delta_j}} \equiv \frac{a}{T_0^{\delta_1}} \quad (27)$$

$$a \neq 0, \quad p_k(z_0) \neq 0.$$

Using the first integral

$$T_0^{1-\delta_1} - \sum_{k=0}^{n} \frac{1 - \delta_1}{1 + k} p_k(z) W_0^{k+1} = C,$$

we arrive at the equation

$$\left(\frac{dW_0}{dt}\right)^{\delta_1 - 1} = a^{\delta_1 - 1} \left[\sum_{k=0}^{n} \frac{1 - \delta_1}{1 + k} p_k(z_0) W_0^{k+1} + C\right]^{\delta_1}.$$

Figure 8: Figure 8

For the absence of moving critical singular points it is necessary [1] $(n+1)\delta_1 \leqslant 2\delta_1 - 2$, but this cannot be for $n > 0$.

Lemma 4. For the system of the form (11) not to contain moving critical singular points, it is necessary that $\frac{\partial Q_2}{\partial x} = 0$.

Let lemma 4 hold, i.e.

$$\frac{dx}{dz} = \sum_{k=0}^{n} p_k(z) y^k, \quad \frac{dy}{dz} = \sum_{k=0}^{N} q_k(z) x^k. \tag{28}$$

If $n = N = 1$, then the system is linear and, therefore, does not contain moving singular points in general.

Let $nN \neq 1$. Introduce the parameter λ:

$$x = \frac{1}{\lambda^{n+1} T}, \quad y = \frac{1}{\lambda^{N+1} W}, \quad z = z_0 + \lambda^{Nn-1} t,$$

for $\lambda = 0$ we obtain

$$\frac{dT_0}{dt} = -\frac{T_0^2}{W_0^n} p_n(z_0), \quad \frac{dW_0}{dt} = -\frac{W_0^2}{T_0^N} q_N(z_0). \tag{29}$$

Consider the one-parameter family of solutions of system (29)

$$T_0 = A(t+C)^{\frac{n+1}{Nn-1}}, \quad W_0 = B(t+C)^{\frac{N+1}{Nn-1}}, \tag{30}$$

where $AB \neq 0$ — constants; C — arbitrary parameter. For the uniqueness of T_0 and W_0 it is necessary to require that $\frac{n+1}{Nn-1} = s$, $\frac{N+1}{Nn-1} = l$, where s and l — integers, obviously $s > 0$, $l > 0$ and $\frac{1}{s} + \frac{1}{l} + \frac{1}{sl} = Nn - 1$. It is easy to see that the latter is possible only for the following values of n and N:

1. $n = 2$, $N = 1$ $(n = 1, N = 2)$.
2. $n = 3$, $N = 1$ $(n = 1, N = 3)$.
3. $n = N = 2$.

Let us have $n = 2$, $N = 1$, i.e.

$$\frac{dx}{dz} = p_2(z) y^2 + p_1(z) y + p_0(z), \quad \frac{dy}{dz} = q_1(z) x + q_0(z), \tag{31}$$

$$p_2(z) \neq 0, \quad q_1(x) \neq 0.$$

This system can be reduced by a linear transformation to the form

$$\frac{dx}{dz} = \widetilde{p}_2(z) y^2 + \widetilde{p}_1(z) y, \quad \frac{dy}{dz} = x, \tag{32}$$

obviously, (32) is equivalent to the equation of the second order

$$\frac{d^2y}{dz^2} = \widetilde{p}_2(z) y^2 + \widetilde{p}_1(z) y. \tag{33}$$

Figure 9: Figure 9

Theorem 3. Equation (33), and therefore, system (32) do not contain movable critical singular points if and only if either $\sim \tilde{p}_2(z)$ and $p_1(z)$ are constant, or by the transformation $y = \mu(z)W + v(z)$, $\tau = \varphi(z)$, where

$$\mu(z) = C\tilde{p}_2^{-1/5}(z), \quad \varphi(z) = \int_{z_0}^{z} \tilde{p}_2^{-2/5}(z)\,dz,$$
$$v = \frac{1}{50\tilde{p}_2^{-3}(z)}\left[6\tilde{p}_2'^{-2}(z) - 5\tilde{p}_2''(z)\tilde{p}_2(z) - 25\tilde{p}_1(z)\tilde{p}_2^{-2}(z)\right], \tag{34}$$

equation (33) reduces to

$$\frac{d^2W}{d\tau^2} = 6W^2 + a\tau + b,$$

where a and b, C are constant numbers. In this cave the solution of system (32) is expressed in terms of elementary, elliptic functions and Painlevé transcendents (solutions of the first Painlevé equation).

Theorem 3'. For the equation (33), and therefore, the system (32) not to contain movable critical singular points when $\tilde{p}_1 \neq$ const, $\tilde{p}_2 \neq$ const, it is necessary and sufficient

$$-\tilde{p}_1(z) + \frac{v''(z)}{v(z)} - \tilde{p}_2(z)v(z) + \frac{\left(a\dfrac{C}{\sqrt{6}}\displaystyle\int_{z_0}^{z}\tilde{p}_2^{-2/5}(z)\,dz + b\right)}{6v(z)}\tilde{p}_2^{-3/5}(z) = 0,$$

where a and b are certain constant numbers[1]).

Theorem 3' follows from Theorem 3 and the form of equation (33) after transformation (34).

Lyct we have $n = 3$, $N = 1$, i.e.

$$\frac{dx}{dz} = p_3(z)y^3 + p_2(z)y^2 + p_1(z)y + p_0(z), \quad \frac{dy}{dz} = q_1(z)x + q_0(z), \tag{35}$$

(35) can be reduced to the form

$$\frac{dx}{dz} = \tilde{p}_3(z)y^3 + \tilde{p}_2(z)y^2 + \tilde{p}_1(z)y, \quad \frac{dy}{dz} = x, \tag{36}$$

which is equivalent to the equation

$$\frac{d^2y}{dz^2} = \tilde{p}_3(z)y^3 + \tilde{p}_2(z)y^2 + \tilde{p}_1(z)y. \tag{37}$$

Analogously to the case $n = 2$, $N = 1$ we have:

Theorem 4. *Equanion (37), and therefore, system (36) do not contain movable critical singular points if and only if either $\tilde{p}_3$, $\sim\tilde{p}_2$, $\sim\tilde{p}_1$ are constant, or by the transformation*

$$y = \mu(z)W + v(z), \quad \tau = \varphi(z), \tag{38}$$

[1]) If $a \neq 0$, then by the choice of z_0 it is possible to ensure $b = 0$, and by the choice of $C - a = 1$.

Figure 10: Figure 10