ON ZOLOTAREV STABILITY OF FUNCTIONALS

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Abstract

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MATHEMATICS

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ON ZOLOTAREV STABILITY OF FUNCTIONALS

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Let $Z_n(x, \theta)$ denote the polynomials determined by the segment-functional $0_0, 0_1, \dots, 0_{n-2}, 1, \theta$ for $(n-1)/2 \le \theta \le (n+1)/2$ (the critical interval). They all belong to the passport $[n, n, 0]$, and the nature of their deformation with respect to θ has been studied in detail in ([1]). Recall that, under a monotone increase of θ, all interior nodes (extremum points) shift to the right. The polynomials $\pm Z_n(x, \theta)$ exhaust all polynomials of the indicated passport.

Definition. The segment-functional $\mu_0, \dots, \mu_{n-1}, \theta$ possesses **Zolotarev stability** if, in the critical interval $\mu_n' \le \theta \le \mu_n''$, the segment belongs to the passport $[n, n, 0]$, and moreover is serviced either by all $+Z_n(x, \theta)$, or by all $-Z_n(x, \theta)$ (and only by them).

Theorem 1. *Whatever the prescribed basis $\mu_0, \mu_1, \dots, \mu_{n-2}$, there always exist two numbers $A_0' \le A_0''$ such that a segment of the form $\mu_0, \dots, \mu_{n-2}, A, \bar{\theta}$ is stable for $A \ge A_0''$ and for $A \le A_0'$. In the first case it is serviced by $Z_n(x, \theta)$, and in the second by $-Z_n(x, \theta)$.*

Expand $\mu_0, \mu_1, \dots, \mu_{n-2}, A$ with respect to the nodes $[\sigma_i(\theta)]_1^n$ of any of the polynomials $\pm Z_n(x, \theta)$ ([1]); we have

$$\delta_j = R_{n-1}^{(j)}(\bar{\mu}, \theta) \,/\, R_n'(\sigma_j) \quad (j = 1, 2, \dots, n), \tag{1}$$

where

$$R_n(x) = \prod_1^n (x - \sigma_i); \qquad R_{n-1}^{(j)}(x) = R_n(x)/(x - \sigma_j).$$

The signs of the denominator in (1) alternate with j; to obtain $\operatorname{Sgn} \delta_j = \pm Z_n(\sigma_j, \theta)$ it is necessary and sufficient that either $R_{n-1}^{(j)}(\bar{\mu}) \ge 0$ $(j = 1, \dots, n)$, or $R_{n-1}^{(j)}(\bar{\mu}) \le 0$ $(j = 1, \dots, n)$ (zeros are also possible). If

$$R_{n-1}^{(j)}(x,\theta) = x^n - s_1^{(j)}(\theta)x^{n-1} + s_2^{(j)}(\theta)x - \cdots,$$

we have

$$R_{n-1}^{(j)}(\bar{\mu},\theta) = A - s_1^{(j)}(\theta)\mu_{n-2} + s_2^{(j)}(\theta)\mu_{n-3} - \cdots + (-1)^{n-2}s_{n-1}^{(j)}(\theta)\mu_0 = A - M_j(\theta).$$

The family of continuous functions $[M_j(\theta)]_1^n$ on $(n-1)/2 \le \theta \le (n+1)/2$ is bounded. Let

$$A_0'' = \max_{(j,\theta)} M_j(\theta) \quad \text{and} \quad A_0' = \min_{(j,\theta)} M_j(\theta);$$

then for fixed $A \ge A_0''$ or $A \le A_0'$ the (δ_i) have alternating signs. The parameter $\bar{\theta}$ for each θ is determined uniquely from the condition $R_n(\bar{\mu}) = 0$, i.e.

$$\bar{\theta} - s_1(\theta)A + s_2(\theta)\mu_{n-2} - \cdots + (-1)^n s_n(\theta)\mu_0 = 0$$

for all $(n-1)/2 \le \theta \le (n+1)/2$. This proves the theorem.

Corollary 1. *For $A_0' < A < A_0''$, the sign-alternation conditions for (δ_j) are not satisfied for the whole family $+Z_n(x,\theta)$ (or $-Z_n(x,\theta)$); that is, the segment does not possess Zolotarev stability.*

Thus, for every basis $\mu_0, \mu_1, \dots, \mu_{n-2}$ there exists a completely determined Zolotarev "critical" interval (A_0', A_0''); then and only then is the segment $\mu_0, \dots, \mu_{n-2}, \mu_{n-1}, \bar{\theta}$ stable, if μ_{n-1} is chosen outside or on the boundary of the interval (A_0', A_0'').

Remark 1. $A_0' = A_0''$ if and only if the basis $(\mu_k)_0^{n-2} \equiv 0$, since here the simultaneous possibility (for $A = A_0' = A_0''$) of two expansions with opposite signs is required.

Corollary 2. Introduce the following notation: $A''(\theta) = \max_{(j)} M_j(\theta)$ and $A'(\theta) = \min_{(j)} M_j(\theta)$. Then the necessary and sufficient condition that the segment $\mu_0, \dots, \mu_{n-2}, \mu_{n-1}, \bar{\theta}$ should under no $\bar{\theta}$ belong to the passport $[n, n, 0]$, i.e. should be completely "detached" from this passport, is the following: $\mu_{n-1} - A''(\theta) < 0$, $\mu_{n-1} - A'(\theta) > 0$ for all θ in the interval $[(n-1)/2, (n+1)/2]$.

Remark 2. If the segment $\mu_0, \dots, \mu_{n-2}, \mu_{n-1}, \theta_0$ is served by some $Z_n(x, \theta_0)$, then the segment $\mu_0, \dots, \mu_{n-2}, \mu_{n-1} + A, \bar{\theta} + A\theta_0$ is served by the same polynomial for any $A > 0$.

Indeed, the segment $0_0, \dots, 0_{n-2}, 1, \theta_0$ is served by the polynomial $Z_n(x, \theta_0)$; consequently, the segment $0_0, \dots, 0_{n-2}, A, A\theta_0$ is also served, and then so is the termwise sum $\mu_0, \dots, \mu_{n-2}, \mu_{n-1} + A, \bar{\theta} + A\theta_0$. Thus, serving by one polynomial is extended in two parameters.

Theorem 2. The Zolotarev interval (A_0', A_0'') of any basis $\mu_0, \mu_1, \dots, \mu_{n-2}$ contains the critical (Chebyshev) interval of the parameter μ_{n-1}.

Thus, if μ_{n-1}', μ_{n-1}'' are the endpoints of the critical interval for the variable parameter μ_{n-1}, then

$$A_0' \leqslant \mu_{n-1}' < \mu_{n-1}'' \leqslant A_0''. \tag{2}$$

Indeed, the segment

$$\mu_0, \dots, \mu_{n-2}, \mu_{n-1}'' + h \tag{3}$$

for $h > 0$ is served by the polynomial $T_{n-1}(x) \equiv Z_n(k, n/2)$ with all $\delta_i \neq 0$. By the theorem on continuous deformation (1), if $\bar{\theta}^*(h)$ is the best continuation of (3) to the n-th place, then the segment $\mu_0, \dots, \mu_{n-2}, \mu_{n-1}'' + h, \bar{\theta}^*(h) \pm \varepsilon$ is served by some $Z_n(x, n/2 \pm \varepsilon)$. But if we take $\mu_0, \dots, \mu_{n-2}, \mu_{n-1}'' - h$ $(h > 0)$, then $T_{n-1}(x)$ is no longer suitable; consequently, some $Z_n(x, n/2 \pm \varepsilon)$ also will not be suitable. Hence $\mu_{n-1}'' - h = A < A_0''$ for every $h > 0$. Therefore, $A_0'' \geqslant \mu_{n-1}''$.

The left-hand part of inequality (2) is proved in the same way.

Corollary. A segment of the form $\mu_0, \dots, \mu_{n-2}, \mu_n, \bar{\theta}$ has "partial Zolotarev stability in the interval $\bar{\theta}^* \leqslant \bar{\theta} \leqslant \bar{\theta}$."

Let us give some applications of the results obtained.

Example 1. The segment-functional

$$1, \rho, \dots, \rho^{n-1}, \rho^n + \bar{\theta} \tag{4}$$

for $\rho > 1$ is Zolotarev-stable, since here $A - M_j(\theta) = R_{n-1}^{(j)}(\rho, \theta) > 0$, i.e. the segment determines all $Z_n(x, \theta)$. For $0 < \rho < 1$ the segment $(\rho^i)_0^{n-1}$, when expanded with respect to arbitrary nodes $(\sigma_i)_1^n$ $(0 \leqslant \sigma_i \leqslant 1$ and $\sigma_i \neq q)$, always gives one repetition of sign for the loads (δ_i). Consequently, the segment (4) is completely detached from the passport $[n, n, 0]$. For $\rho = 1$ the segment (4) is served over the whole critical interval only by Chebyshev transforms $T_n(ax)$, i.e. although (4) belongs only to the passport $[n, n, 0]$, the requirement of completeness is not fulfilled—the serving is partial, but without involving extraneous passports.

Example 2. The segment-functional $(\mu_i)_0^n = 0_0, \dots, 0_{n-3}, 1_{n-2}, n/2, \bar{\theta}$ gives an example of partial serving by Zolotarev polynomials, with the involvement of polynomials of another passport. Indeed, since here $n/2 = \mu_{n-1}''$, the basis $(\mu_i)_0^{n-1}$ is served by the polynomial $T_{n-1}(x)$

with one unloaded end node $\sigma_1 = 0$. Let us find the boundaries of the Zolotarev critical interval from the conditions of Theorem 1, which here take the form

$$A_0'' = \max_{j,\theta}[\theta - \sigma_j(\theta)]$$

and

$$A_0' = \min_{(j,\theta)}[\theta - \sigma_j(\theta)].$$

For all θ in the interval $[(n-1)/2,\ (n+1)/2]$ one has

$$\theta - \sigma_n(\theta) \le \theta - \sigma_{n-1}(\theta) \le \dots \le \theta - \sigma_1(\theta).$$

Thus, the max is attained at $\theta - \sigma_1(\theta)$, and the min at $\theta - \sigma_n(\theta)$. Finally we have

$$A_0'' = (n+1)/2 - \tilde{\sigma}_1 (> n/2);$$
$$A_0' = (n-1)/2 - \tilde{\sigma}_{n-1} (< n/2 - 1),$$

where $\tilde{\sigma}_1$ and $\tilde{\sigma}_{n-1}$ are the corresponding nodes of

$$T_n(x) = \cos n \arccos(2x-1).$$

Thus, the segment $(\mu_i)_0^n$ is served only by part of the polynomials $Z_n(x,\theta)$, and only in part of the critical interval $\bar{\theta}^* \le \bar{\theta} \le \theta''$. In the remaining part, service belongs to another passport.

Example 3. The segment $(\mu_i)_0^n = \alpha_0, \alpha_1, \dots, \alpha_{n-1}, \bar{\theta}$ with the amorphous basis $(\alpha_i)_0^{n-1}$ cannot, for any $\bar{\theta}$, be served by primitive Zolotarev polynomials, i.e. by those among whose nodes $(\sigma_i)_1^n$ there are both $\sigma_1 = 0$ and $\sigma_n = 1$. Indeed, if for some $\bar{\theta}_0$ one has

$$\alpha_k = \sum_1^{s_1} \delta_i' \sigma_{i'}^k - \sum_1^{s_2} \delta_i'' \sigma_i''^k \quad (k = 0, 1, \dots, n-1),$$

then the resulting equalities

$$\alpha_k + \sum_1^{s_1} \delta_i'' \sigma_i''^k = \sum_1^{s_1} \delta_i' \sigma_{i'}^k \quad (k = 0, 1, \dots, n-1)$$

are impossible, since on the left we have an amorphous segment, and on the right a nodal one ([2]).

In this note the question of Zolotarev stability is studied from a somewhat different standpoint than in the article ([3]).

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[2] E. V. Voronovskaya, DAN, **166**, No. 6 (1966).
[3] E. V. Voronovskaya, DAN, **161**, No. 2 (1965).

Note: Figure translations are in progress. See original paper for figures.