A method for introducing the finite parts of divergent integrals, and applications of them in operational calculus

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Abstract

A number of problems in mathematical physics lead to the consideration of divergent integrals. Cauchy and Hadamard developed an algorithm that allows assigning a well-defined finite part to certain divergent integrals. This article proposes a method for introducing the finite part of singular functions and divergent integrals, and examines the inclusion of certain classes of non-integrable functions into operational calculus. Bibliography: 7 items.

Full Text

Preamble

This work follows the foundational developments established in 1967 by V. A. Ditkin [1, 2] and subsequent refinements in operational calculus [4]. We begin by defining the core functional spaces and operators used throughout this study.

§ 1. Basic Definitions and Properties

Let $\phi(x)$ be a function defined such that as $x \to 0$, $\phi(x)$ behaves asymptotically according to a reference function $\psi(x)$. We consider the behavior of $\phi(x)$ in the interval $0 < x < \delta$. For a given function $f(x)$, we define the limit at the origin as:

$$\lim_{x \to 0} [f(x) - \phi(x)] = A$$

where $\phi(x) \in \mathfrak{R}$. If there exists another representation such that:

$$\lim_{x \to 0} [f(x) - \phi_1(x)] = B$$

where $\phi_1(x) \in \mathfrak{R}$, then it follows from the properties of the space $\mathfrak{R}$ that $A = B$.

The following algebraic properties hold for these limits: 1. Linearity: $|f(0) + g(0) = |f(0) + |g(0)$ 2. Scalar Multiplication: $|cf(0) = c|f(0)$ 3. Product Rule: If $a(x)$ is a sufficiently smooth function, then $|a(0)f(0) = a(0)\phi(0) + a(0)|f(0)$.

To prove the product rule, consider the limit:

$$\lim_{x\to 0}[a(x)f(x) - \psi(x)] = \lim_{x\to 0}[a(x)f(x) - a(x)\phi(x)] + \lim_{x\to 0}[a(x)\phi(x) - \psi(x)]$$

Given that $\lim a(x) = a(0)$ and $\lim[f(x) - \phi(x)] = |f(0)$, we obtain the stated result.

For a point $x = a > 0$, we define the left-hand and right-hand limits by shifting the argument. Let $a_1(x) = f(a+x)$ and $a_2(x) = f(a-x)$ for $0 < x < \delta$. Then the limits at $x = a$ are denoted as $|f(a+0)$ and $|f(a-0)$. If $|f(a+0) = |f(a-0)$, the function is considered continuous at $x = a$ in the sense of Ditkin and Prudnikov.

§ 2. Integral Operators and Convolution

Let $f(t)$ be a function defined on $0 < \delta < t < T$. We define the integral operator $F(x) = \int f(\xi)d\xi$ and associate it with the space $\mathfrak{R}$. The following properties are established for these integral representations: 1. Additivity: $\int[f(t) + g(t)]dt = \int f(t)dt + \int g(t)dt$ 2. Integration by Parts: For functions $f(t)$ and $g(t)$ in $\mathfrak{R}$, the formula for integration by parts is given by:

$$\int f'(t)g(t)dt = f(t)g(t) - |f(0)g(0) - \int f(t)g'(t)dt$$

3. Jump Discontinuities: If $f(t)$ has a jump at $x = a > 0$, the derivative $f'(t)$ in the operational sense accounts for the discontinuity:

$$\int f'(t)dt = f(t) - f(0) - [|f(a+0) - |f(a-0)]$$

§ 3. Volterra Integral Equations

Consider the integral equation of the second kind:

$$R(t) = \int_0^t K(t-u)f(u)du$$

where $K(t)$ is the kernel and $f(u)$ is the unknown function. Using the methods described in [6, 7], we can express the solution using the resolvent kernel $Q(t)$. If $Q(t) \in M$, then the solution $R(t)$ belongs to the space L.

The relationship between the kernels $K(t)$ and $H(t)$ can be expressed via the convolution:

$$\int_0^t H_1(t-v)R(v)dv = \int_0^t K_1(t-v)L(v)dv$$

where H_1 and K_1 are the integrated kernels. This allows for the systematic solution of equations where the kernel is a power function or an exponential.

§ 4. Applications and Examples

1. **Power Kernel:** Let $K(t) = t^n$. The resulting integral $R(t) = \int_0^t (t-u)^n f(u)du$ can be evaluated using Gamma functions. For $n > \alpha$, we obtain:
$$R(t) = \frac{\Gamma(1-\alpha)\Gamma(n+1)}{\Gamma(n-\alpha+2)}(t-a)^{n-\alpha+1}$$
2. **Logarithmic Singularities:** In cases where the kernel involves logarithmic terms, such as those arising from the constant $C = 0.5772$ (Euler-Mascheroni constant), the operational calculus provides a streamlined approach to evaluating the limits as $t \to a$.
3. **Heaviside Step Functions:** Using the notation $\eta(t)$ for the unit step function, we can represent piecewise continuous kernels. For example, a kernel defined as $K(t) = \eta(t) - \eta(t-a)$ leads to solutions involving shifted arguments, which are common in control theory and signal processing.

The results presented here extend the classical operational methods of Cauchy and are consistent with the modern theory of generalized functions as presented in [5, 6].

Note: Figure translations are in progress. See original paper for figures.

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