LOCALLY NILPOTENT RADICAL OF MAL' TSEV ALGEBRAS SATISFYING THE \(n\)-TH ENGEL CONDITION

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Abstract

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LOCALLY NILPOTENT RADICAL OF MAL' TSEV ALGEBRAS SATISFYING THE n-TH ENGEL CONDITION

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Higman' s theorem on the nilpotency of solvable Lie algebras satisfying the n-th Engel condition ([1]) turns out to be valid for Mal' tsev algebras with a finite number of generators (Theorem 1). However, in contrast to Lie algebras, the condition that the number of generators be finite is essential here, as the corresponding example shows. It is further proved that in a Mal' tsev algebra satisfying the n-th Engel condition there exists a maximal locally nilpotent ideal, which is called the radical. For Lie algebras this result was obtained by A. I. Kostrikin ([2]). Just as in the case of Lie algebras, the radical constructed has the radical property: the quotient algebra by it is semisimple in the sense of this radical.

Mal' tsev algebras, which naturally generalize the class of Lie algebras, were first defined in ([3]) as anticommutative algebras satisfying the identity

$$[(xy)z]x + [(yz)x]x + [(zx)x]y = (xy)(xz). \tag{1}$$

As Sagle showed ([4]), over a field of characteristic different from 2, in a Mal' tsev algebra A the identity

$$[(xy)z]t + [(yz)t]x + [(zt)x]y + [(tx)y]z = (ty)(xz), \tag{2}$$

also holds, which for $t = x$ turns into (1). Relation (2) means that in the multiplication algebra $\bar{A}$ of the algebra A relations of the form

$$R_{xR_{yR}}z = R_{zR_{xR}}y + R_{yR}x + R_{yR_{xz}} + R_{(xy)z}. \tag{3}$$

hold.

Remark. From relation (3) it follows that any word of length $2k$ in the generators of the algebra A can be represented in the form of a linear combination of right-normalized products of elements of A, each of which contains at least $k+1$ factors.

Put $A^{(0)} = A$, $A^{(k)} = A^{(k-1)}A^{(k-1)}$ $(k \geq 1)$. Just as in ([5]), by A^k we shall denote the subalgebra in A generated by all possible words of length k in the generators of the algebra A. If $A^k = 0$ $(A^{(k)} = 0)$ for some $k > 0$, then the algebra A is called **nilpotent** (**solvable**). The notions of **local nilpotency** and **local solvability** are defined in the natural way. An algebra A is said to **satisfy the n-th Engel condition** (condition E_n) if for every $x \in A$, $R_x^n = 0$. Everywhere in what follows we shall assume the characteristic of the ground field to be different from 2.

Theorem 1. *Let the Mal' tsev algebra A be solvable and satisfy condition E_n. Then the algebra A is locally nilpotent.*

The assertion of global nilpotency under the hypotheses of Theorem 1 is, generally speaking, false.

Example. Let Γ be the Grassmann algebra with a countable set of generators $\{e_i\}$ and with the standard basis consisting of associative words $e_{i_1}e_{i_2}\dots e_{i_k}$ $(i_1 < i_2 < \dots < i_k;\ k \geq 1)$. On the linear space Γ define an ant-

anticommutative multiplication $x \cdot y$, setting, for the basis elements Γ,

$$(e_{i_1}e_{i_2}\dots e_{i_k}) \cdot e_j = e_{i_1}e_{i_2}\dots e_{i_k}e_j, \qquad k \geq 1;$$

$$(e_{i_1}e_{i_2}\dots e_{i_k}) \cdot (e_{j_1}e_{j_2}\dots e_{j_m}) = 0, \qquad k, m \geq 2.$$

It is easily verified that the algebra $A = \Gamma(+,\cdot)$ is a Mal' cev algebra, satisfies the condition E_3, and $A^{(2)} = 0$. At the same time the algebra A, obviously, is not nilpotent.

For the proof of Theorem 1 we need some auxiliary definitions and assertions. The following lemma is a simple consequence of identity (2).

Lemma 1. *Let I be an ideal of a Mal' cev algebra A. Then $I^2 + I^2A$ is also an ideal in A.*

For an arbitrary ideal I in A define a chain of ideals $I_k = L_k(I)$, putting $I_0 = I$, $I_k = I_{k-1}^2 + I_{k-1}^2A$ $(k \geq 1)$; $I_0 \supset I_1 \supset \cdots$. It is clear that $L_k(I_m) = I_{k+m}$. We shall call the ideal I **L-solvable** if $I_k = 0$ for some $k > 0$. Since $I_k \supset I^{(k)}$, every L-solvable ideal of the Mal' cev algebra A is solvable. The converse, however, is also true.

Lemma 2. *Every solvable ideal of a Mal' cev algebra A is L-solvable.*

Proof. We shall show that $I_2 \subset I^2$. It is clearly enough to prove that

$$I_1^2A = (I^2 + I^2A)^2A \subset I^2.$$

In turn, this reduces to proving the inclusions $(I^2 \cdot I)A \subset I^2$ and $[(I^2A)I] \subset I^2$, which follow from identity (2). Suppose it has already been proved that $I_{2k} \subset I^{(k)}$. Then

$$I_{2k+2} = L_2(I_{2k}) \subset I_{2k}^2 \subset I^{(k+1)}.$$

Lemma 3. *Let A be a Mal' cev algebra with a finite set of generators $\{a_1, a_2, \dots, a_l\}$, satisfying the condition E_n. Then for every $k \geq 0$ there exists a number $N(k)$ such that for all $m \geq N(k)$,*

$$A^m \subset A_k = L_k(A).$$

Proof. For $k = 0$ the assertion is trivial; suppose it has already been proved for $k-1$. From the nonassociative words in the generators $a_1, \dots, a_l$ construct a basis of the free anticommutative algebra with l generators. Order this basis $\mathcal{E} = \{u_1, u_2, \dots\}$ arbitrarily, assuming that words of smaller length are less than words of greater length. Since this will cause no misunderstanding, the u_j may also be regarded as elements of the algebra A; for the linear space A they form a set of generators. Let

$$R = R_{c_1} R_{c_2} \cdots R_{c_z} \qquad (c_i \in A),$$

where each of the elements c_i is equal to one of the elements u_j. We shall call a subword $R_{c_{i-1}} R_{c_i} R_{c_{i+1}}$ **regular** if $c_{i-1} \leq c_i$, $c_{i-1} < c_{i+1}$, or $c_{i-1} = c_i = c_{i+1}$, and **irregular** otherwise. Using relations (3), R may be represented as a linear combination of analogous expressions R_j that contain no irregular subwords. Let $\mathcal{E}_{k-1}$ be the set of all words from $\mathcal{E}$ whose length is less than $N(k-1)$, and let M be the total length of the words from $\mathcal{E}_{k-1}$. Consider a word

$$c = (c_1 c_2) \dots c_m \in A,$$

where $m \geq N(k-1) + (n-1)M + 1$ and each of the elements c_i is equal to one of the elements u_j. Denote

$$c' = (c_1 c_2) \dots c_{N(k-1)};$$

then $c' \in A_{k-1}$, $c = c'R$ $(R \in \bar{A})$, and c is represented as a linear combination of elements

$$c' R_{v_1} R_{v_2} \dots R_{v_t},$$

where $v_1 \leq v_2 \leq \cdots$. The total length of the elements $v_1, v_2, \dots$, considered as words in the generators $a_1, \dots, a_l$, is not less than

$$m - N(k-1) > (n-1)M.$$

Taking into account that A satisfies the condition E_n, we conclude from this that at least one of the elements

$$v_1, v_2, \dots \notin \mathcal{E}_{k-1}.$$

By the induction hypothesis, this element belongs to A_{k-1}; hence $c \in A_k$. Thus, for $N(k)$ one may take the number

$$2[N(k-1) + (n-1)M].$$

Theorem 1 follows immediately from Lemmas 2 and 3.

Lemma 4. *If a Malʹ cev algebra A satisfies the condition E_n and has a finite number of generators, then the algebra $A^{(k)}$ $(k \geq 1)$ also has a finite number of generators.*

Proof. We shall show that the algebra $A^{(1)} = A^2$ has a finite number of generators. Let $\mathcal{E}'$ be the set of elements of $\mathcal{E}$ whose length is greater than 1 and less than $N(4)$. By Lemma 3, every word in the generators of the algebra A whose length is $\geq N(4)$ lies in A_4 and, by Lemma 2, in $A^{(2)}$, i.e., it is represented as a linear combination of pairwise products of elements from $A^{(1)}$ having smaller length. Consequently, to obtain a set of generators of $A^{(1)}$ it suffices to take the set $\mathcal{E}'$. The rest is obvious.

From Theorem 1 and Lemma 4 the following theorem follows.

Theorem 2. *If I is an ideal of a Malcev algebra A satisfying the condition E_n, and both I and A/I are locally nilpotent, then the algebra A is also locally nilpotent.*

Theorem 3. *If A is a Malcev algebra satisfying the condition E_n, and B, C are locally nilpotent ideals of A, then $B + C$ is also a locally nilpotent ideal of A.*

Proof. It suffices to consider the algebra $B + C/C \simeq B/B \cap C$ and apply Theorem 2.

By virtue of Theorem 3, in Malcev algebras satisfying the condition E_n there exists a maximal locally nilpotent ideal (coinciding with the maximal locally solvable ideal), which it is natural to call the radical of these algebras (cf. ([2])).

Theorem 4. *Let A be a Malcev algebra with condition E_n, and let K be the radical of A. Then the algebra A/K is semisimple, i.e., it has no locally nilpotent ideals.*

Proof. Indeed, let B' be a locally nilpotent ideal of the algebra $A' = A/K$. Then its inverse image B under the homomorphism $A \to A'$ will be an ideal of the algebra A, satisfying the conditions of Theorem 2. Consequently, B is a locally nilpotent ideal of A and strictly contains K, which is impossible.

Finally, note that all the results of the present paper carry over without any changes to arbitrary Σ-operator rings satisfying the identities $x^2 = 0$ and (2).

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Note: Figure translations are in progress. See original paper for figures.