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Abstract

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MATHEMATICS

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A SUMMATION THEOREM AND ITS APPLICATION TO SPECIAL FUNCTIONS

(Presented by Academician Yu. V. Linnik, 28 X 1966)

This article gives representations in the form of convergent series for several special functions: the gamma-, psi-, and zeta-functions, as well as hypergeometric functions. The basis of the work is a summation formula analogous to the Euler–Maclaurin identity.

Theorem 1. *Let $f(z)$ be an analytic function in the entire complex plane except for a set M of singular points. Then*

$$\Delta z \sum_{i=1}^{n} f(\xi + (i-1)\Delta z) =$$

$$= \int_{z_1}^{z_2} f(z)\, dz + \sum_{j=1}^{\mu} (-1)^j \frac{B_j(\theta)}{j!} \Delta z^j \left[f^{(j-1)}(z_2) - f^{(j-1)}(z_1) \right] +$$

$$+ \sum_{j=\mu+1}^{\infty} A_{j\mu}(\theta) \Delta z^{j+1} \sum_{i=1}^{n} f^{(j)}(\xi + (i-1)\Delta z), \qquad \mu = 0, 1, \dots, \tag{1}$$

where $z_1 = \xi + (\theta - 1)\Delta z$, $z_2 = \xi + (n + \theta - 1)\Delta z$; the complex numbers $\Delta z, \xi, \theta$ need only satisfy the inequality

$$\max\{|\theta|,\ |\theta - 1|\} < \min_{i=0,1,\dots,n-1} \inf_{z \in M} \left| \frac{z - \xi}{\Delta z} - i \right|;$$

$B_j(\theta)$ are the Bernoulli polynomials; $B_j = B_j(0)$ are the Bernoulli numbers;

$$A_{j\mu}(\theta) = \sum_{\nu=0}^{\mu} (-1)^\nu \frac{B_\nu(\theta)}{\nu!} \frac{(\theta-1)^{j-\nu+1} - \theta^{j-\nu+1}}{(j-\nu+1)!} =$$

$$= \frac{(-1)^\mu}{(j-\mu)!\mu!} \sum_{\nu=0}^{\mu} B_\nu C_\mu^\nu \frac{(-1)^\nu (\theta-1)^{j-\nu+1} - \theta^{j-\nu+1}}{j-\nu+1};$$

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the integral is computed along the straight line joining the limits of integration, or along any curve obtained from it by a continuous deformation without crossing points of the set M.

Theorem 1 is proved by induction on μ, using the recurrence formula for Bernoulli polynomials

$$(-1)^j \frac{B_j(\theta)}{j!} = \sum_{\nu=0}^{j-1} (-1)^\nu \frac{B_\nu(\theta)}{\nu!} \frac{(\theta-1)^{j-\nu+1} - \theta^{j-\nu+1}}{(j-\nu+1)!}.$$

Consequences of formula (1) are expansions of the gamma-function, the psi-function, and the Riemann zeta-function.

Theorem 2. *The following expansion of the gamma-function holds for every* z *lying outside the circles of radius* $1/2$ *with centers at the negative integers:*

$$\Gamma(1+z) = \sqrt{2\pi}\,(z+\theta)^{z+1/2} \exp\left\{ -(z+\theta) + \sum_{j=1}^{\mu} \frac{B_{j+1}(\theta)}{j(j+1)(z+\theta)^j} + \right.$$
$$\left. + \sum_{j=\mu+1}^{\infty} D_{j\mu}(\theta)\zeta(j,z) \right\}, \qquad \mu = 0, 1, \dots, \tag{2}$$

where

$$\zeta(j,z) = \sum_{i=1}^{\infty} \frac{1}{(z+i)^j}$$

is the generalized Riemann zeta-function;

$$D_{j\mu}(\theta) = \frac{(-1)^{j-\mu+1} C_j^{\mu+1}}{j} \sum_{\nu=0}^{\mu+1} B_\nu C_{\mu+1}^{\nu} \frac{(-1)^\nu (\theta-1)^{j-\nu+1} - \theta^{j-\nu+1}}{j-\nu+1},$$

and the range of values of the arbitrary parameter θ is determined by the inequality

$$\max\{|\theta|, |\theta-1|\} < \min_{i=1,\dots,\infty} |z+i|,$$

in which case the series in the exponent converges absolutely.

The proof of the expansion (2) shows that the gamma-function is uniquely determined by its asymptotic behavior at infinity as $\operatorname{Re} z \to \infty$ and by the functional equation $\Gamma(1+z) = z\Gamma(z)$.

If θ is regarded as an arbitrary given number, then the expansion (2) holds for any z lying outside circles with centers at negative integers and radius equal to $\max\{|\theta-1|,|\theta|\}$.

In (^{1}) special cases of equality (2) are given: for $\theta=0$ and $\mu=0$ we obtain Binet' s formula, and for $\theta=1/2$ and $\mu=0$, Burnside' s formula.

Theorem 3. Under the conditions of Theorem 2 the following expansion of the psi-function into an absolutely convergent series holds

$$\psi(1+z)=\ln(z+\theta)-\sum_{j=1}^{\mu}\frac{B_j(\theta)}{j(z+\theta)^j}-$$

$$-\sum_{j=\mu+1}^{\infty} jD_{j,\mu-1}(\theta)\zeta(j+1,z), \qquad \mu=0,1,\ldots. \tag{3}$$

From (3) we find an expansion for Euler' s constant

$$C=-\psi(1)=-\ln\theta+\sum_{j=1}^{\mu}\frac{B_j(\theta)}{j\theta^j}+\sum_{j=\mu+1}^{\infty} jD_{j,\mu-1}(\theta)\zeta(j+1)$$

for arbitrary nonnegative integer μ and any θ lying in the region $\max\{|\theta|,|\theta-1|\}<1$. For $\mu=0$, for example, one may write

$$C=\sum_{i=1}^{\nu}\frac{1}{i}-\ln(\nu+\theta)+\sum_{j=2}^{\infty}(-1)^{j+1}\frac{(\theta-1)^j-\theta^j}{j}\zeta(j,\nu),$$

where $\nu=0,1,\ldots$ and θ is any value in the region $\max\{|\theta|,|\theta-1|\}<\nu+1$. The most advantageous value for rapid convergence is $\theta=1/2$.

Theorem 4. For any $\nu=0,1,\ldots;\ \mu=0,1,\ldots$ and any complex $s\neq 1$,

$$\zeta(s)=\sum_{i=1}^{\nu}\frac{1}{i^s}+\frac{(\nu+\theta)^{1-s}}{\Gamma(s)}\sum_{j=0}^{\mu}\frac{B_j(\theta)\Gamma(s-1+j)}{j!(\nu+\theta)^j}+$$

$$+\frac{1}{\Gamma(s)}\sum_{j=\mu+1}^{\infty}(-1)^{jA}_{j\mu}(\theta)\Gamma(s+j)\left[\zeta(s+j)-\sum_{i=1}^{\nu}\frac{1}{i^{s+j}}\right], \tag{4}$$

where the series converges absolutely for arbitrary θ in the region

$$\max\{|\theta|,|\theta-1|\}<\nu+1.$$

A particular case ($\nu=1,\ \mu=0,\ \theta=0$) of formula (4) is known (^{2}). The expansion of the gamma function (^{2}) makes it possible to write a representation of the Stirling numbers (3) of the first kind $C_{i+1}^{(k)}$ and of the second kind $\mathfrak{C}_i^k$ in terms of Bernoulli polynomials:

$$C_{i+1}^{(k)} = \sum \frac{f_1^{\nu_1}(-i) \dots f_k^{\nu_k}(-i)}{\nu_1! \dots \nu_k!}, \qquad \mathfrak{C}_i^k = \sum \frac{f_1^{\nu_1}(i) \dots f_k^{\nu_k}(i)}{\nu_1! \dots \nu_k!},$$

$$f_j(i) = \frac{B_{j+1}(i) - B_{j+1}}{j(j+1)},$$

where the sum is taken over all solutions of the equation $\nu_1 + 2\nu_2 + \dots + k\nu_k = k$ in nonnegative integers $\nu_1, \dots, \nu_k$.

Recall that the Stirling numbers are defined by the equalities

$$x(x+1)\dots(x+i) = \sum_{k=0}^{i} C_{i+1}^{(k)} x^{\,i-k+1},$$

$$\frac{1}{x(x+1)\dots(x+i-1)} = \sum_{k=0}^{\infty} \frac{(-1)^k \mathfrak{C}_i^k}{x^{i+k}} \qquad (x > n-1).$$

If $C_{i+1}^{(k)}$ and $\mathfrak{C}_i^k$ are regarded as polynomials of degree $2k$ in i, then $C_{i+1}^{(k)} = \mathfrak{C}_{-i}^k$ and $\mathfrak{C}_i^k = C_{-i+1}^{(k)}$, i.e., it is sufficient to have an expression in the form of a polynomial in i for the Stirling numbers, for example of the first kind. By induction one can show that

$$C_{i+1}^{(k)} = (i+1)i \dots (i-k+1) \sum_{j=0}^{k-1} g_{kj}(i-k)(i-k-1)\dots(i-k-j+1), \tag{5}$$

where the coefficients g_{kj} are found from the recurrence relation

$$g_{kj} = \frac{(k+j)g_{k-1,j} + g_{k-1,j-1}}{k+j+1}, \qquad g_{k0} = \frac{1}{k+1}, \qquad g_{k,k-1} = \frac{1}{2^k k!}.$$

Writing $C_{i+1}^{(k)}$ in the form of a factorial polynomial is convenient because in the recurrence relation for the coefficients only three neighboring coefficients turn out to be connected.

Representation (5) makes it possible immediately to write the expansion of the confluent hypergeometric function $\Phi(a, c; z)$ in Bessel functions.

Theorem 5. For arbitrary complex $a \neq 0$, $c \neq 0, -1, -2, \dots$, $z \neq 0$,

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$$\frac{\Phi(a,c;z)}{\Gamma(c)\,\left(i\sqrt{az}\right)^{1-c}} =$$

$$= J_{c-1}(2i\sqrt{az}) + az\sum_{k=1}^{\infty}\left(-i\sqrt{\frac{z}{a}}\right)^{k}\sum_{j=0}^{k-1} g_{kj}\left(-i\sqrt{az}\right) J_{c+k+j}(2i\sqrt{az})\,. \qquad (6)$$

The scheme of the proof is as follows: in the series defining $\Phi(a,c;z)$, one must replace $a(a+1)\ldots(a+i-1)$ by its expression in terms of the Stirling numbers of the first kind, change the order of summation, replace the Stirling numbers by their representation (5), once again change the order of summation, and use the representation of the Bessel functions in the form of a series. We note that formula (6) is close to Tricomi' s expansion (^{4}).

Similarly, one can expand the hypergeometric function $F(a,b;c;z)$ in values of the confluent hypergeometric function.

Theorem 6. In the circle $|z| < 1$, for arbitrary complex $a \neq 0$, b, $c \neq 0, -1, -2, \ldots$

$$F(a,b;c;z) = \Phi(b,c;az) + \\ + \sum_{k=1}^{\infty}\frac{1}{a^k}\sum_{j=k+1}^{2k}\frac{d(b+1)\ldots(b+j-1)}{c(c+1)\ldots(c+j-1)}\,g_{k,k-j-1}(az)^j\Phi(b+j,c+j;az). \qquad (7)$$

Expansions (6) and (7) are especially effective when $|az|$ is bounded and $a \to \infty$.

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REFERENCES

1. G. Bateman, A. Erdélyi, *Higher Transcendental Functions. Hypergeometric Function. Legendre Functions*, Moscow, 1965.

2. E. C. Titchmarsh, *The Theory of the Riemann Zeta-function*, Oxford, 1951.

3. N. Nielsen, *Handb. d. Theorie der Gammafunktion*, 1906.

4. F. Tricomi, *Ann. Mat. Pura et Appl.* (4), **28**, 263 (1949).

Note: Figure translations are in progress. See original paper for figures.

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