FOURIER SERIES AND MAXIMAL THEOREMS

MATHEMATICS

1967

SovietRxiv

Abstract

Full Text

UDC 517.51

MATHEMATICS

L. V. ZHIZHIASHVILI

FOURIER SERIES AND MAXIMAL THEOREMS

(Presented by Academician I. N. Vekua on 8 VII 1966)

Consider a function of two variables $f(x,y)$. Suppose that it is periodic with respect to each of the variables and that $f(x,y) \in L(R)$, where $R = [-\pi,\pi;-\pi,\pi]$. Denote by $\bar{f}_i(x,y)$ $(i = 1,2,3)$ the conjugate functions of two variables

$$\bar{f}_1(x,y) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+s,y)\operatorname{ctg}\frac{s}{2}\,ds,$$

$$\bar{f}_2(x,y) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x,y+t)\operatorname{ctg}\frac{t}{2}\,dt,$$

$$\bar{f}_3(x,y) = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x+s,y+t)\operatorname{ctg}\frac{s}{2}\operatorname{ctg}\frac{t}{2}\,dt\,ds.$$

By $\sigma[f]$ we shall denote the double Fourier series of the function $f(x,y)$, and by the symbols $\bar{\sigma}[f;x]$, $\bar{\sigma}[f;y]$, and $\bar{\sigma}[f;x,y]$ the double conjugate trigonometric series to the series $\sigma[f]$, respectively in the variable x, in the variable y, and in the totality of the variables x and y. Further, let $\sigma_{mn}^{(i,\alpha,\beta)}(x,y)$ $(i = 0,1,2,3)$, $\alpha,\beta > 0$, be the Cesàro $(C;\alpha,\beta)$-means respectively of the series $\sigma[f]$, $\bar{\sigma}[f;x]$, $\bar{\sigma}[f;y]$, $\bar{\sigma}[f;x,y]$, and put

$$\varphi_i(x,y) = \sup_{m,n\geq 0}\left|\sigma_{mn}^{(i,\alpha,\beta)}(x,y)\right| \qquad (i = 0,1,2,3),$$

$$\varphi_4(x,y) = \sup_{0<\varepsilon,\eta\leq\pi}\left|\int_{\varepsilon}^{\pi}\int_{\eta}^{\pi}\frac{f(x+s,y+t)-f(x-s,y+t)-f(x+s,y-t)+f(x-s,y-t)}{\operatorname{tg}s/2\ \operatorname{tg}t/2}\,dt\,ds\right|,$$

$$\varphi_5(x,y) = \sup_{0<\varepsilon\leq\pi}\left|\int_{\varepsilon}^{\pi}\frac{f(x+s,y)-f(x-s,y)}{\operatorname{tg}s/2}\,ds\right|,$$

$$\varphi_6(x,y) = \sup_{0<\eta\le\pi}\left|\int_\eta^\pi \frac{f(x,y+t)-f(x,y-t)}{\operatorname{tg} t/2}\,dt\right|.$$

Further, if $s_{ij}(x,y)$ $(i,j=0,1,\dots)$ denote the partial sums of the series $\sigma[f]$, then the expressions

$$\sigma_n^\alpha(x,y) = \sum_{k=0}^{n} A_{n-k}^{\alpha-1} s_{kk}(x,y), \qquad \alpha>0,$$

will be called the (C,α)-means of the series $\sigma[f]$.

In the present note we give assertions concerning questions of the existence of the function $\bar f_3(x,y)$ and the summability of the functions $\varphi_i(x,y)$ $(i=0,\dots,6)$; we also study the behavior of the (C,α)-means of the series $\sigma[f]$.

As A. Zygmund showed ([1]), if $f(x,y)\in L\log^+ L$, then the function $\bar f_3(x,y)$ exists almost everywhere. In ([2]) he also posed the question: if $f(x,y)\in$

$\in L(R)$, then does there exist almost everywhere the function $f_3(x,y)$, where

$$\bar f_3(x,y) = \lim_{(\varepsilon,\eta)_\lambda\to 0}\int_\varepsilon^\pi\int_\eta^\pi \frac{f(x+s,y+t)-f(x-s,y+t)-f(x+s,y-t)+f(x-s,y-t)}{\operatorname{tg} s/2+\operatorname{tg} t/2}\,dt\,ds, \tag{1}$$

where the symbol $(\varepsilon,\eta)_\lambda\to 0$ means that $\varepsilon\to 0$, $\eta\to 0$, and $1/\lambda\le\varepsilon/\eta\le\lambda$, $\lambda\ge 1$. E. Stein (see ([3]), Theorem 5) showed that, in contrast to the case of a function of one variable, the limit (1) may fail to exist on a set of positive planar measure even in the case when $\lambda=1$.

Below we shall show that the scheme of the proof (see ([3]), Theorem 5) given by E. Stein contains an error. To verify this, we shall need the following

Lemma. Let $(x,y)\in[0,1/3;0,1/3]\equiv I$ and $\varepsilon>0$. Then, if

$$E^\varepsilon(\alpha)\equiv E(\alpha) = \left\{(x,y)\in I : \frac{1}{xy|\ln x|^\varepsilon|\ln y|^\varepsilon}\ge\alpha\right\},$$

then for every $\alpha\ge\alpha_0(\varepsilon)>0$ the measure

$$\operatorname{mes} E(\alpha) \ll \begin{cases} \dfrac{A}{\alpha}(\ln\alpha)^{1-2\varepsilon}, & \text{if } 1-2\varepsilon>0,\\ \dfrac{A'}{\alpha}, & \text{if } 1-2\varepsilon<0,\end{cases}$$

where A and A' are certain positive constants.

Proof. Put

$$E_1(\alpha) = \left\{(x, y) \in I : \frac{1}{xy|\ln x|^\varepsilon|\ln y|^\varepsilon} \geq \alpha,\ y \leq x\right\},$$

$$E_2(\alpha) = \left\{(x, y) \in I : \frac{1}{xy|\ln x|^\varepsilon|\ln y|^\varepsilon} \geq \alpha,\ y > x\right\},$$

$$E_3(\alpha) = \left\{(x, y) \in I : \frac{1}{xy|\ln x|^{2\varepsilon}} \geq \alpha\right\},$$

$$E_4(\alpha) = \left\{(x, y) \in I : \frac{1}{xy|\ln y|^{2\varepsilon}} \geq \alpha\right\}.$$

It is not difficult to see that $E(\alpha) \subset E_1(\alpha) + E_2(\alpha) \subset E_3(\alpha) + E_4(\alpha)$. Consequently, the measure

$$\operatorname{mes} E(\alpha) \leq \operatorname{mes} E_3(\alpha) + \operatorname{mes} E_4(\alpha). \tag{2}$$

But the set

$$E_3(\alpha) = \left\{(x, y) \in I :\ y \leq \frac{1}{\alpha x|\ln x|^{2\varepsilon}}\right\}. \tag{3}$$

Let the number $x_0 \equiv x_0(\alpha, \varepsilon)$ be chosen so that $1/\alpha x_0|\ln x_0|^{2\varepsilon} = 1/3$. It is clear that

$$1/\alpha^2 < x_0 < 1/\alpha, \qquad \alpha \geq \alpha_0(\varepsilon). \tag{4}$$

Thus, taking into account (3) and (4), we shall have

$$\operatorname{mes} E_3(\alpha) \ll \frac{1}{\alpha}\int_{1/\alpha^2}^{1/3} \frac{dx}{x|\ln x|^{2\varepsilon}} + \frac{1}{\alpha} \ll \begin{cases} \dfrac{A}{2\alpha}(\ln\alpha)^{1-2\varepsilon}, & 1 - 2\varepsilon > 0, \\ A'/2\alpha, & 1 - 2\varepsilon < 0;\ \alpha \geq \alpha_0(\varepsilon). \end{cases} \tag{5}$$

Similarly,

$$\operatorname{mes} E_4(\alpha) \ll \begin{cases} \dfrac{A}{2\alpha}(\ln\alpha)^{1-2\varepsilon}, & 1 - 2\varepsilon > 0, \\ A'/2\alpha, & 1 - 2\varepsilon < 0,\ \alpha \geq \alpha_0(\varepsilon). \end{cases} \tag{6}$$

From (2), (5), and (6) the validity of the lemma follows.

Analyzing the proof of the lemma, it is not difficult to see that if $1-2\varepsilon=0$, i.e. $\varepsilon=1/2$, then

$$\operatorname{mes} E(\alpha) \ll \frac{A''}{\alpha} \ln\ln\alpha, \qquad \alpha \geq \alpha_0(\varepsilon).$$

Let us note here also that the following estimate is valid:

$$\operatorname{mes} E(\alpha) \geqslant \frac{B}{\alpha}(\ln\alpha)^{1-2\varepsilon}, \quad 1-2\varepsilon>0,\ \alpha \geqslant \alpha_0(\varepsilon), \tag{7}$$

where B is a positive constant. But the proof of relation (7) is more complicated than the proof of the lemma, and therefore we do not give it here.

The assertion that

$$\operatorname{mes} E(\alpha) \geqslant B'/\alpha, \quad 1-2\varepsilon<0, \quad \alpha \geqslant \alpha_0(\varepsilon), \quad B'>0,$$

is, generally speaking, false. Thus, according to the lemma and (7), we shall have

$$\operatorname{mes} E(\alpha) \sim (\ln\alpha)^{1-2\varepsilon}/\alpha, \quad 1-2\varepsilon>0 \quad (\alpha\to\infty).$$

Now, analyzing the proof of E. Stein (see [3], Theorem 5, pp. 159-160), one can conclude that the function $f_\delta(x,y)$ belongs to the class $L(\log L)^{1-\varepsilon}$ ($\varepsilon>0$, arbitrarily small) only when $\delta>1-\varepsilon>1/2$; that is, $1-2\delta<0$, and, according to the lemma,

$$\operatorname{mes} E^\delta(\alpha) \leqslant A'/\alpha, \quad \alpha \geqslant \alpha_0(\delta),$$

which contradicts assertion (13.6) of E. Stein (see [3], p. 160). Consequently, E. Stein (see [3], p. 154) obtains no contradiction, and thereby he has not proved that if $f(x,y)\in L(\log L)^\alpha$ for some $\alpha\in[0,1)$, then, generally speaking, the double conjugate series $\bar{\sigma}[f;x,y]$ is not summable almost everywhere by the Poisson method. Consequently, Stein' s conclusion on the nonexistence of $\bar{f}_3(x,y)$, made in the conclusion and referring to Theorem 5, is insufficiently convincing.

We now give results connected with questions of summability of the functions $\varphi_i(x,y)$ $(i=0,\dots,6)$.

Theorem 1. If $f(x,y)\in L\log L$, then

$$\varphi_i(x,y)\in L^q(R) \quad \text{for all } q\in(0,1) \quad (i=0,\dots,4). \tag{8}$$

If, however, $f(x,y) \in L(\log L)^2$, then

$$\varphi_i(x,y) \in L(R) \quad (i = 0, \dots, 4). \tag{9}$$

Let us note that for functions $f(x,y) \in L(\log L)^\alpha$ for all $\alpha \in [0,1)$, assertion (8) is, generally speaking, false; and for the validity of assertion (9) the condition $f(x,y) \in L(\log L)^2$ is also essential, since the following theorem is true.

Theorem 2. There exists a nonnegative 2π-periodic function $f_0(x,y) \in L(\log L)^{2-\varepsilon}$ for all $\varepsilon \in (0,2]$, however $\varphi_i(x,y) \notin L(R)$ $(i = 0, \dots, 4)$.

For the summability of the functions $\varphi_i(x,y)$ $(i = 5,6)$, the condition $f(x,y) \in L \log L$ is sufficient, which follows from a known theorem of A. Zygmund [4].

Theorem 3. Let $f(x,y) \in L(R)$. If $\bar{f}_i(x,y)$ $(i = 1,2)$ are summable, then the series $\bar{\sigma}[f;x]$ and $\bar{\sigma}[f;y]$ are Fourier series respectively of the functions $\bar{f}_1(x,y)$ and $\bar{f}_2(x,y)$; but if $f(x,y) \in L \log L$ and $\bar{f}_3(x,y) \in L(R)$, then the series $\bar{\sigma}[f;x,y]$ is also the Fourier series of the function $\bar{f}_3(x,y)$.

J. Marcinkiewicz [5] proved that if $f(x,y) \in L \log L$, then almost everywhere $\sigma_n^1(x,y) \to f(x,y)$ as $n \to \infty$. In fact, the following is true.

Theorem 4. If $f(x,y) \in L(R)$, then for every $\alpha > 0$, almost everywhere

$$\lim_{n\to\infty} \sigma_n^\alpha(x,y) = f(x,y).$$

The author expresses deep gratitude to Prof. P. L. Ulyanov for valuable advice.

Received
8 VII 1966

REFERENCES

[1] A. Zygmund, Fund. Math., **36**, 207 (1949).
[2] A. Zygmund, Rend. Mat. e applic., **17**, fasc. 2–4, 468 (1957).
[3] E. M. Stein, Ann. Math., **74**, No. 1, 140 (1961).
[4] A. Zygmund, Fund. Math., **13**, 284 (1929).
[5] J. Marcinkiewicz, Ann. Scuola norm. super Pisa, **8**, 149 (1939).

Note: Figure translations are in progress. See original paper for figures.