ON THE EXTENSION OF THE VARIATIONAL METHOD OF G. M. GOLUZIN–P. P. KUFAREV TO MULTIPLY CONNECTED DOMAINS

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Abstract

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I. A. ALEKSANDROV, A. S. SOROKIN

ON THE EXTENSION OF THE VARIATIONAL METHOD OF G. M. GOLUZIN–P. P. KUFAREV TO MULTIPLY CONNECTED DOMAINS

(Presented by Academician M. A. Lavrent' ev, 9 XI 1966)

The aim of the present work is to extend the well-known variational theorem of G. M. Goluzin ([1]) to finitely connected domains. Earlier P. P. Kufarev, jointly with N. V. Genina (Semukhina) ([2]) (see also ([5])), using a new integral representation of the first variation given by him ([3]), extended G. M. Goluzin' s theorem to families of univalent functions in an annulus. Important results in the same direction were obtained, on the basis of other considerations, by M. Schiffer ([6]) and S. A. Gel' fer ([7]). From the main variational theorem (Theorem 4) of our work there follows the above-mentioned theorem of P. P. Kufarev. The formula indicated in Theorem 1, which generalizes Schwarz' s formula to the case of finitely connected circular domains, is a development and strengthening of the results of V. A. Zmorovich ([8]) and G. Meshkovskii ([9]). Starting from this formula and using a number of considerations of P. P. Kufarev ([3]), we obtain theorems on differentiability with respect to the parameter t of families of functions $F(w,t)$ and $\Phi(z,t)$, regular in a finitely connected domain, a variational theorem, a generalized Löwner equation, and also a variational formula for a finitely connected domain of the type of the variational formula of M. A. Lavrent' ev for the disk.

Let the real numbers $R_k > 0$ and the complex numbers a_k ($a_0 = a_n = 0$), $k = 0, \dots, n$, be such that the inequalities $|a_i| < R_0 - R_i$, $|a_i - a_j| > R_i + R_j$, $i, j = 1, \dots, n$, are satisfied.

Theorem 1. Let $f(z)$ be a function regular and single-valued inside the intersection of the disk $|z| < R_0$ with the exteriors of the disks $|z - a_k| \le R_k$, $k = 1, \dots, n$, whose real part on the boundary components $C_k:\ |\zeta - a_k| = R_k$, $k = 0, \dots, n$, assumes the values $f_k(\zeta)$. Then the formula holds

$$f(z) = \sum_{k=0}^{n} \frac{\delta_k}{2\pi i} \int_{C_k} f_k(\zeta) H_k(\zeta, z) \frac{d\zeta}{\zeta}$$

$$-\frac{1}{4\pi i}\int_{C_0} f_0(\zeta)\,\frac{d\zeta}{\zeta}-\frac{1}{4\pi i}\int_{C_n} f_n(\zeta)\,\frac{d\zeta}{\zeta}+iD,$$

where D is a real number, $\delta_0=1,\ \delta_k=-1,\ k=1,\dots,n,$

$$H_k(\zeta,z)=\frac{\zeta+z}{\zeta-z}+\sum_{\nu=1}^{\infty}\left[\sum_{k_1}^{2\nu}{}' p(\zeta,z;k_1,k_2,\dots,k_{2\nu})+\sum_{k_1}^{2\nu-1} p(\zeta,z;k,k_1,\dots,k_{2\nu-1})\right],$$

and

$$p(\zeta,z;k_1,k_2,\dots,k_{2\nu})=$$

$$=\left[\frac{A_{k_{2\nu}}z}{R^2_{k_{2\nu}}t_{2\nu-1}+a_{k_{2\nu}}t_{2\nu}}+\frac{A_{h-k_{2\nu}}}{t_{2\nu}}\right]\frac{\zeta}{R^2_{k_{2\nu}}t_{2\nu-1}-(z-a_{k_{2\nu}})t_{2\nu}}\prod_{i=1}^{2\nu}R^2_{k_i},$$

$$A_k=1+\frac{1}{2}(\delta_k-\delta_{n-k}),\qquad \sum_{k_1}^{k_{2\nu}}{}'=\sum_{\substack{k_1=0\\k_1\neq k}}^{n}\ \sum_{\substack{k_2=0\\k_2\neq k_1}}^{n}\cdots\sum_{\substack{k_{2\nu}=0\\k_{2\nu}\neq k_{2\nu-1}}}^{n}.$$

The functions t_p are determined by the recurrence relations

$$t_p=-b_{p-1}t_{p-1}+R^2_{k_{p-1}}t_{p-2},\quad t_0=1,\quad t_1=\zeta-a_{k_1},$$

where $b_p=a_{k_{p+1}}-a_{k_p}$, if p is even, and $b_p=\bar a_{k_{p+1}}-\bar a_{k_p}$, if p is odd.

The functions $f_k(\zeta)$ satisfy the conditions

$$\frac{1}{2\pi i}\int_{C'_m} f_m(\zeta)\frac{d\zeta}{\zeta-a_m}=\sum_{\substack{k=0\\k\neq m}}^{n}\frac{\delta_k}{2\pi i}\int_{C'_k} f_k(\zeta)S_k(\zeta)\frac{d\zeta}{\zeta},\qquad m=1,\dots,n,$$

where

$$S_k(\zeta)=T_k[\zeta,\bar L_m(\infty)]+\frac{\delta_k-1}{2}\,T_k(\zeta,\infty),$$

$$T_k(\zeta,z)=\frac{1}{2}H_k(\zeta,z)-\frac{\zeta L'_k(\zeta)}{2L_k(\zeta)}\,\bar H_k[L_k(\zeta),\bar z],\qquad L_k(\zeta)=\bar a_k+\frac{R_k^2}{\zeta-a_k}.$$

Theorem 2. Let a family of finitely $(n+1)$-connected domains $G(w,t)$, $a\le t\le b$, be given in the w-plane, with the following properties:

1) the boundary components $C_k(t)$, $k = 0, \dots, n$, of the domain $G(w, t)$ are simple closed Jordan curves $w = \Omega_k(\lambda, t)$, where $0 \le \lambda \le 2\pi$, $k = 0, \dots, n$, and none of the $C_k(t)$, $k = 1, \dots, n$, lies inside another, while all of them lie inside $C_0(t)$.
2) the functions $\Omega_k(\lambda, t)$, $k = 0, \dots, n$, are differentiable in t, uniformly with respect to λ, at $t = t_0$, $t_0 \in [a, b]$.
3) the curves $C_k(t_0) : w = \Omega_k(\lambda, t_0)$, $k = 0, 1, \dots, n$, are analytic.

Further, let $z = F(w, t)$ be a function mapping conformally $G(w, t)$ onto the intersection of the disk $|z| < R_0(t)$ with the exteriors of the disks $|z - a_k| \le R_k(t)$, $k = 1, \dots, n$, where the functions $R_k(t)$, $k = 0, \dots, n$, are differentiable at the point $t = t_0$; let there exist

$$\frac{\partial}{\partial t}[\arg F(w, t)]\Big|_{t=t_0}, \qquad R_k(t_0) = R_k.$$

Then $F(w, t)$ is differentiable in t, uniformly with respect to w inside $G(w, t_0)$, at $t = t_0$, and for the derivative $F_t'(w, t_0)$ the formula holds

$$\frac{\partial F(w, t_0)}{\partial t} = -F(w, t_0) \left\{ \frac{1}{4\pi i} \int_{|\zeta| = R_0} L_0(\zeta) \frac{d\zeta}{\zeta} + \frac{1}{4\pi i} \int_{|\zeta| = R_n} L_n(\zeta) \frac{d\zeta}{\zeta} - iD \right.$$

$$\left. - \sum_{k=0}^{n} \frac{\delta_k}{2\pi i} \int_{|\zeta - a_k| = R_k} L_k(\zeta) H_k[\zeta, F(w, t_0)] \frac{d\zeta}{\zeta} \right\},$$

where

$$L_k(\zeta) = -\frac{\partial}{\partial t} \ln\left|F[\Omega_k(\lambda, t), t_0]/R_k(t)\right|\Big|_{t=t_0}.$$

In this case the relations

$$\frac{1}{2\pi i} \int_{|\zeta - a_m| = R_m} L_m(\zeta) \frac{d\zeta}{\zeta - a_m} = \sum_{\substack{k=0 \\ k \ne m}}^{n} \frac{\delta_k}{2\pi i} \int_{|\zeta - a_k| = R_k} L_k(\zeta) S_k(\zeta) \frac{d\zeta}{\zeta},$$

$$m = 1, \dots, n. \tag{*}$$

Theorem 3. Under the conditions of Theorem 2, the function $w = \Phi(z, t)$, inverse to $F(w, t)$, is differentiable in t, uniformly with respect to z inside the intersection of the disk $|z| < R_0(t)$ with the exteriors of the disks $|z - a_k| \le R_k(t)$, $k = 1, \dots, n$, at $t = t_0$, and

$$\left.\frac{\partial\Phi(z,t)}{\partial t}\right|_{t=t_0} = z\frac{\partial\Phi(z,t_0)}{\partial z}\left\{\frac{1}{4\pi i}\int_{|\zeta|=R_0} L_0(\zeta)\frac{d\zeta}{\zeta} + \frac{1}{4\pi i}\int_{|\zeta|=R_n} L_n(\zeta)\frac{d\zeta}{\zeta}\right.$$

$$\left. -\sum_{k=0}^{n}\frac{\delta_k}{2\pi i}\int_{|\zeta-a_k|=R_k} L_k(\zeta)H_k(\zeta,z)\frac{d\zeta}{\zeta} - iD\right\}.$$

where

$$L_k(\xi) = \frac{d}{dt}\ln R_k(t_0) - \operatorname{Re}\left[\frac{1}{\xi\Phi'_\zeta(\xi,t)}\frac{\partial}{\partial t}\Omega_k(\lambda,t)\right]_{t=t_0}.$$

In this case the relations $(*)$ are satisfied.

If in Theorems 2 and 3 one sets $n=0$ and $n=1$, then the well-known theorems of P. P. Kufarev ([2–4]) are obtained.

Theorem 4. Let a function $f(z)$ be holomorphic and univalent in the intersection of the disk $|z|<R_0$ with the exteriors of the disks $|z-a_k|\le R_k,\ k=1,\dots,n$. Suppose that the function $f_k(z,t)=f(z)+tg_k(z)$ is holomorphic and univalent for each $t\in[0,T]$ in the annulus $R_k<|z-a_k|\le R_k+\varepsilon_k$ for small $\varepsilon_k>0,\ k=1,\dots,n$, and maps it onto a domain with boundary continua $C_{R_k}(t)$ and $C_k(t)$. Further, let $f_0(z,t)=f(z)+tg_0(z)$, for each $t\in[0,T]$, be holomorphic and univalent in the annulus $R_0-\varepsilon_0\le|z|<R_0,\ \varepsilon_0>0$, and map this annulus onto a domain with boundary continua $C_0(t)$ and $C_{R_0}(t)$. We shall assume that each $C_{R_k}(t)$ lies inside $C_k(t)$, $k=1,\dots,n$, $C_{R_0}(t)$–outside $C_0(t)$, none of the curves $C_k(t)$, $k=1,\dots,n$, lies inside another, and, at the same time, all of them lie inside $C_0(t)$. Let the functions $R_k(t)$, $k=0,\dots,n$, be differentiable at the point $t=0$, $R_k(0)=R_k$. Then the function $\Phi(z,t)$, conformally and univalently mapping the domain $K(t)$, which is the intersection of the disk $|z|<R_0(t)$ with the exteriors of the disks $|z-a_k|\le R_k(t)$, $k=1,\dots,n$, onto the $(n+1)$-connected domain $G(w,t)$, whose boundary consists of the continua $C_{R_k}(t)$, $k=0,\dots,n$, is representable inside $K(t)$ in the form

$$\Phi(z,t)=f(z)+tzf'(z)P(z)+o(t),$$

where

$$P(z)=\sum_{k=0}^{n}\frac{\delta_k}{2\pi i}\lim_{\rho_k\to R_k}\int_{|\zeta-a_k|=\rho_k}\operatorname{Re}B_k(\zeta)H_k(\zeta,z)\frac{d\zeta}{\zeta}$$

$$-\frac{d}{dt}\ln\sqrt{R_0(0)R_n(0)}-iD-\frac{1}{4\pi i}\lim_{\rho_0\to R_0}\int_{|\zeta|=\rho_0}\operatorname{Re}B_0(\zeta)\frac{d\zeta}{\zeta}$$

$$-\frac{1}{4\pi i}\lim_{\rho_n\to R_n}\int_{|\zeta|=\rho_n}\operatorname{Re}B_n(\zeta)\frac{d\zeta}{\zeta},\qquad B_k(\zeta)=\frac{g_k(\zeta)}{\zeta f'(\zeta)}.$$

In this case

$$\sum_{\substack{k=0\\k\neq m}}^{n}\beta_k\frac{d}{dt}\ln R_k(0)-\frac{d}{dt}\ln R_m(0)=\sum_{\substack{k=0\\k\neq m}}^{n}\frac{\delta_k}{2\pi i}\lim_{\rho\to R_k}\int_{|\zeta-a_m|=\rho}\operatorname{Re}B_k(\zeta)S_k(\zeta)\frac{d\zeta}{\zeta}$$

$$-\frac{1}{2\pi i}\lim_{\rho\to R_m+0}\int_{|\zeta-a_m|=\rho}\frac{\operatorname{Re}B_m(\zeta)\,d\zeta}{\zeta-a_m},\qquad m=1,\dots,n,$$

$$\beta_k=\frac{\delta_k}{2\pi i}\lim_{\rho\to R_k}\int_{|\zeta-a_k|=\rho}S_k(\zeta)\frac{d\zeta}{\zeta}.$$

If in Theorem 4 one sets $n=1$, then the theorem of P. P. Kufarev and N. V. Genina (Semukhina) ([2,5]) is obtained. If, then, R_1 is allowed to tend to zero, we obtain the theorem of P. P. Kufarev ([4]).

Theorem 5. Let a family of finite $(n+1)$-connected domains $G(w,t)$, $t\in[\tau_0,0]$, be such that:

1) $w_0\in G(w,t)$;
2) $G(w,t_1)\subset G(w,t_2)$ for $t_1>t_2$; $t_1,t_2\in(\tau_0,0]$;
3) the boundary components $C_k(t)$, $k=0,\dots,n$, of the domain $G(w,t)$ are simple closed Jordan curves $\Omega_k(\theta,t)$, uniformly with respect to θ, $0\le\theta\le 2\pi$, differentiable with respect to t on $(\tau_0,0]$.

Then the function $w=\Phi(z,t)$, $\Phi(z_0,t)=w_0$, mapping the intersection of the disk $|z|<R_0(t)$ with the exterior of the disks $|z-a_k|\leqslant R_k(t)$, $k=1,\dots,n$, onto the domain $G(w,t)$, satisfies at $(t_0,0)$ the equation

$$\Phi_t(z,t)\big|_{t=t_0}=z\Phi_z(z,t_0)\left\{\frac{1}{4\pi}\Psi_0(2\pi,t_0)+\frac{1}{4\pi}\Psi_n(2\pi,t_0)-iD-\sum_{k=0}^{n}\frac{\delta_k}{2\pi}\int_0^{2\pi}H_k[a_k+R_k(t)e^{i\theta},z]\,d\Psi_k(\theta,t_0)\right\},$$

where

$$\Psi_k(\theta,t_0)=-\lim_{\rho_k=1}\int_0^{\theta}\frac{\partial}{\partial t}\ln\left|\frac{F\{\Phi[a_k+\rho_kR_k(t)e^{i\theta},t],t_0\}}{R_k(t)}\right|_{t=t_0}d\theta,$$

$$\Psi_m(2\pi, t_0) = \sum_{\substack{k=0 \\ k\neq m}}^{n} \lim_{\rho_k \to 1} \delta_k \int_0^{2\pi} B_k[\theta, \rho_k R_k(t_0)]\, d\Psi_k(\theta, t_0), \qquad m = 1, \dots, n,$$

where

$$B_k(\theta, \rho) = -\frac{\rho e^{i\theta}}{a_k + \rho e^{i\theta}}\, S_k(a_k + \rho e^{i\theta}), \qquad F(w, t) \equiv \Phi^{-1}(z, t).$$

Theorem 5 is a direct generalization of the Löwner-type equation to families of finitely connected domains, obtained earlier for a family of simply connected domains by P. P. Kufarev ([3]), and for doubly connected domains by I. A. Aleksandrov ([10]).

Theorem 6. Let the $(n+1)$-connected domain G have boundary components

$$C_k: \quad w_k(\theta) = a_k + R_k[1 + \sigma_k(\theta)]e^{i\theta}, \qquad 0 \le \theta \le 2\pi, \quad \operatorname{Im} \sigma_k(\theta) = 0,$$

$k = 0, \dots, n$. We shall assume that $|\sigma_k(\theta)| < \varepsilon$, where $\varepsilon > 0$ and is small, and that C_k is star-shaped with respect to a_k. Let $\Phi(z)$ map conformally and univalently the domain K, formed by the intersection of $|z| < R_0$ with the exterior of the disks $|z - a_k| \leqslant R_k$, $k = 1, \dots, n$, onto G. Then, up to small terms of order $o(\varepsilon)$ inside K, the function $\Phi(z)$ is represented in the form

$$\Phi(z) \simeq z \Bigg\{ 1 + \sum_{k=0}^{n} \frac{\delta_k}{2\pi} \int_0^{2\pi} \sigma_k(\theta) H_k(a_k + R_{ke}^{i\theta}, z) \frac{d\theta}{1 + a_{kR} k^{-1} e^{-i\theta}} - \frac{1}{4\pi} \int_0^{2\pi} \sigma_0(\theta)\, d\theta - \frac{1}{4\pi} \int_0^{2\pi} \sigma_n(\theta)\, d\theta - \frac{d}{dt} \ln$$

Moreover,

$$\sum_{\substack{k=0 \\ k\neq m}}^{n} \beta_k \frac{d}{dt} \ln R_k(0) - \frac{d}{dt} \ln R_m(0) =$$

$$= \sum_{\substack{k=0 \\ k\neq m}}^{n} \frac{\delta_k}{2\pi} \int_0^{2\pi} \sigma_k(\theta) B_k(\theta, R_k)\, d\theta - \frac{1}{2\pi} \int_0^{2\pi} \sigma_m(\theta)\, d\theta, \qquad m = 1, \dots, n.$$

From Theorem 6, by simple means, one derives the results obtained earlier for $n = 0$ by M. A. Lavrent' ev ([11]), for $n = 1$ by I. A. Aleksandrov ([10]) and G. V. Siryk ([12]).

Tomsk State University
named after V. V. Kuibyshev

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CITED LITERATURE

[1] G. M. Goluzin, *Mat. sbornik*, **19** (61), 2, 203 (1946).
[2] P. P. Kufarev, *DAN*, **107**, No. 4 (1956).
[3] P. P. Kufarev, *Mat. sbornik*, **13** (55), 1 (1943).
[4] P. P. Kufarev, *DAN*, **97**, No. 3 (1954).
[5] N. V. Genina (Semukhina), *DAN*, **107**, No. 4 (1956); *Uch. zap. TGU*, No. 44, 226 (1962).
[6] M. Schiffer, *Am. J. Math.*, **65**, 341 (1943).
[7] S. A. Gel' fer, *DAN*, **142**, No. 3 (1962).
[8] V. A. Zmorovich, *Dokl. AN USSR*, **5**, 489 (1958).
[9] H. Meschkowski, *Math. Zs.*, **62**, 161 (1955).
[10] I. A. Aleksandrov, *Sibirsk. matem. zhurn.*, **4**, No. 5, 961 (1963).
[11] M. A. Lavrent' ev, B. V. Shabat, *Methods of the Theory of Functions of a Complex Variable*, 1951.
[12] G. V. Siryk, *Izv. vyssh. uchebn. zaved.*, matem., 5 (1960).

Note: Figure translations are in progress. See original paper for figures.