SovietRxiv

Soviet-era science, translated into English

Reports of the Academy of Sciences of the USSR

PHYSICS

1967

SovietRxiv

View the original and related papers at https://sovietrxiv.org/items/ru-196701.92449

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1967. Volume 176, No. 5

UDC 530.12:531.51

PHYSICS

I. S. SHIKIN

A HOMOGENEOUS AXISYMMETRIC COSMOLOGICAL MODEL IN THE ULTRARELATIVISTIC CASE

(Presented by Academician L. I. Sedov on 29 XII 1966)

The present paper is a continuation of ([1]), in which a homogeneous axisymmetric solution of the Einstein equations and Maxwell equations was considered under the condition that space is filled with a perfect fluid and, in the comoving system, there is a homogeneous magnetic field directed along the axis of symmetry z. In the comoving system, which is at the same time synchronous, the metric has the form ([1])

$$-ds^2 = -c^2 d\tau^2 + a^2(\tau)\left[d\chi^2 + f^2(\chi) d\varphi^2\right] + b^2(\tau) dz^2. \tag{1}$$

In the closed model $f(\chi) = \sin\chi$, in the open model $f(\chi) = \operatorname{sh}\chi$, in the flat model $f(\chi) = \chi$.

For dustlike matter $a(\tau)$ and $b(\tau)$ are given in ([1]). Below we consider the case of the ultrarelativistic equation of state $e = 3p$.

The equation $T^k_{0;k} = 0$ for $e = 3p$ gives ([1]) $(2\dot{a}/a) + (\dot{b}/b) = -3\dot{e}/4e$ (the dot denotes differentiation with respect to proper time τ), i.e.

$$e = \frac{3D}{8\pi k/c^4}\,(a^2 b)^{-4/3}, \qquad D = \text{const}. \tag{2}$$

In the case under consideration, the gravitational equations reduce to three equations (([1]), formulas (3)–(5)), of which we shall use the following two together with equation (2):

$$\frac{\dot{a}^2}{c^2 a^2} + 2\frac{\dot{a}\dot{b}}{c^2 ab} \pm \frac{\alpha^2}{a^2} = \frac{8\pi k}{c^4}(e + W) = \frac{3D}{(a^2 b)^{4/3}} + \frac{B_1^2}{a^4}; \tag{3}$$

SovietRxiv

$$\frac{2\ddot{a}}{c^2 a} + \frac{\dot{a}^2}{c^2 a^2} \pm \frac{\alpha^2}{a^2} = \frac{8\pi k}{c^4}(-p + W) = -\frac{D}{(a^2 b)^{4/3}} + \frac{B_1^2}{a^4}. \tag{4}$$

The closed model corresponds in (3)–(4) to the upper sign and $\alpha^2 = 1$, the open model to the lower sign and $\alpha^2 = 1$; for the flat model $\alpha^2 = 0$.

1. Let us consider the solution for the case when there is no magnetic field ($B_1^2 = 0$). This question was considered earlier in ([2], [3]).

For $B_1^2 = 0$, from (3) and (4) we obtain for a and b the equations

$$3Y\frac{d^2Y}{da^2} - 6\left(\frac{dY}{da}\right)^2 \mp 14\alpha^2\frac{dY}{da} - 8\alpha^4 = 0, \qquad Y \equiv \frac{a\dot{a}^2}{c^2}. \tag{5}$$

$$\frac{D}{b^{4/3}} = a^{2/3}\left[\mp\alpha^2 - \frac{dY}{da}\right]. \tag{6}$$

It is convenient to introduce the parameter ξ according to

$$\frac{1}{\xi^2} = \mp\alpha^2 - \frac{d(a\dot{a}^2/c^2)}{da} \equiv \mp\alpha^2 - \frac{dY}{da}. \tag{7}$$

The possibility of introducing the parameter in this way (for physically real conditions, when $e = 3p > 0$ and the constant $D > 0$) follows from formula (6), which then assumes the form

$$b^{4/3} = \frac{D\xi^2}{a^{2/3}}. \tag{8}$$

Equation (5) is transformed with the aid of (7) into the form

$$\frac{d(a\dot{a}^2/c^2)}{(a\dot{a}^2/c^2)} = 3\,\frac{(\mp\alpha^2\xi^2 - 1)\,d\xi}{\xi(3 \mp \alpha^2\xi^2)}. \tag{9}$$

From formulas (7) and (9) we obtain

$$da = (a\dot{a}^2/c^2)\,3\zeta\,d\zeta/(3 + a^2\zeta^2). \tag{10}$$

Integrating (9) and (10), we obtain a and $\dot{a}$ as functions of ζ, after which $\tau(\zeta)$ is also determined. The dependence $b(\zeta)$ is expressed by (8).

Closed model. Let us first consider the case $\zeta^2 = 3$ (ζ then cannot serve as a parameter). From (7), introducing η according to $(2/3^{1/2})c\,d\tau = a\,d\eta$, we obtain

SovietRxiv

$$a = A(1 - \cos\eta), \qquad (2/3^{1/2})c(\tau - \tau_0) = A(\eta - \sin\eta), \qquad A = \text{const}. \quad (11)$$

The curve (11) is a cycloid. $b^4 = (3D)^3/a^2$. For $a = 0$, $b = \infty$ and $e = 3p = \infty$. Excluding the case just considered, from (9) and (10) we obtain

$$a\dot{a}/c^2 = A(\zeta^2 - 3)^2/\zeta, \qquad A = \text{const}; \quad (12)$$

$$a(\zeta)/A = A_1 + 9\zeta - \zeta^3, \qquad A_1 = \text{const}; \quad (13)$$

$$c(\tau - \text{const})/A = \pm 3 \int^{\zeta} [\zeta a(\zeta)/A]^{1/2}\, d\zeta. \quad (14)$$

The range of variation of ζ is determined by the requirement that the expression under the radical, $\zeta a(\zeta)/A$, not be negative. In this case various cases are possible, depending on the magnitude of the constant A_1.

If $A_1 > 6 \cdot 3^{1/2}$, then the cubic (13) has one real root $\zeta = \zeta_1 > 0$; $\zeta a(\zeta)/A \geq 0$ for $0 \leq \zeta \leq \zeta_1$. According to (13), (12), (8), and (2) we have: for $\zeta = \zeta_1$, $a = 0$, $\dot{a} = \infty$, $b = \infty$, and $e = 3p = \infty$; for $\zeta = 0$, $a/A = A_1 \neq 0$, $\dot{a} = \infty$, $b = 0$, and $e = \infty$. Near the value $\zeta = 0$, taking $\tau = \tau_0$ at $\zeta = 0$, we have

$$a/A \approx A_1 + 9(4A_1)^{-1/3}[c(\tau - \tau_0)/A]^{2/3}.$$

It is seen that this formula is valid for $\tau > \tau_0$ and for $\tau < \tau_0$ (on both sides of the value $a/A = A_1,\ e = \infty$). Near $\zeta = \zeta_1$, taking $\tau = \tau_1$ at $\zeta = \zeta_1$, we obtain

$$a/A \approx [27(\zeta_1^2 - 3)^2/12\zeta_1]^{1/3}[c(\tau - \tau_0)/A]^{2/3}.$$

This formula is also valid for $\tau > \tau_1$ and for $\tau < \tau_1$ (on both sides of the value $a = 0,\ e = \infty$). In the interval $2\tau_0 - \tau_1 \leq \tau \leq \tau_1$, the solution is given by (14) with const $= \tau_0$ and with $\zeta = 0$ as the lower limit of integration. To continue the solution through the value $\tau = \tau_1$, in (14) one must take the corresponding additive constant. Thus we obtain a periodic dependence $a(\tau)$; over one period $a(\tau)/A$ increases from the value $a = 0,\ \dot{a} = \infty$ ($b = \infty,\ e = \infty,\ \zeta = \zeta_1 > 3^{1/2}$), passes through a maximum ($\zeta = 3^{1/2}$), and decreases to the value $a/A = A_1,\ \dot{a} = \infty$ ($b = 0,\ e = \infty$), after which it changes in the reverse order down to the value $a = 0$.

For $A_1 = 6 \cdot 3^{1/2}$, from (13) and (14) we obtain, putting $\zeta = 2 \cdot 3^{1/2} \cos^2\lambda$:

$$a/A = 6 \cdot 3^{1/2} \sin^2\lambda\,(2\cos^2\lambda + 1)^2,$$
$$c\tau/A = (27/4)^{1/2}(12\lambda + 2\sin^3 2\lambda - 3\sin 4\lambda). \quad (15)$$

SovietRxiv

For $0 < A_1 < 6 \cdot 3^{1/2}$, the cubic (13) has three real roots $\zeta_1 > 0$, $\zeta_2 < 0$, and $\zeta_3 < \zeta_2 < 0$; $\zeta a(\zeta)/A \geq 0$ for $\zeta_3 \leq \zeta \leq \zeta_2$ and for $0 \leq \zeta \leq \zeta_1$, so that two types of solutions are possible. Solutions with $0 \leq \zeta \leq \zeta_1$ are completely analogous to those considered above. The type of solutions for which $\zeta_3 \leq \zeta \leq \zeta_2$ turns out to be different: for $\zeta = \zeta_3$, $a = 0$, $\dot{a} = \infty$, $b = \infty$, and $e = \infty$; for $\zeta = \zeta_2$ likewise $a = 0$, $\dot{a} = \infty$, $b = \infty$, and $e = \infty$. The function b in this case does not vanish. The asymptotic formulas for $a(\tau)$ near $\zeta = \zeta_3$ and $\zeta = \zeta_2$ are analogous to the one given above for $\zeta = \zeta_1$; the dependence $a(\tau)$ is analytically continued on both sides of the value $a = 0,\ e = \infty$ and has a periodic character.

For $A_1 = 0$, evaluating the integral (14), we obtain

$$(a/A)^2 = \zeta^2(9 - \zeta^2)^2, \qquad c(\tau - \text{const})/A = \mp(9 - \zeta^2)^{3/2}. \tag{16}$$

For $\zeta = 0$, $a = 0$, $\dot{a} = \infty$, $b = 0$, and $e = \infty$; for $\zeta^2 = 9$, $a = 0$, $\dot{a} = \infty$, $b = \infty$, and $e = \infty$.

Negative values of the constant A_1 do not lead to new types of solutions, since the corresponding formulas reduce to the formulas with positive A_1 by changing the sign of ξ and a/A.

Open model. Integrating (9) and (10), we obtain

$$\frac{a\dot{a}^2}{c^2} = A\frac{(\xi^2 + 3)^2}{\xi}, \qquad A = \text{const}; \tag{17}$$

$$a/A = A_1 + 9\xi + \xi^3, \qquad A_1 = \text{const}; \tag{18}$$

$$c(\tau - \tau_0)/A = \pm 3 \int^{\xi} [\xi a/A]^{1/2}\, d\xi. \tag{19}$$

Again the range of variation of ξ is determined by the requirement that $\xi a/A$ be ≥ 0. The trinomial (18), for all values of the constant A_1, has one real root $\xi = \xi_1$. For $A_1 > 0$ this root is $\xi_1 < 0$; $\xi a/A \geq 0$ for $\xi \leq \xi_1$ and for $\xi \geq 0$, so that two types of solutions are possible. The first type corresponds to $\xi \geq 0$. At $\xi = 0$ $(\tau = \tau_0)$, $a/A = A_1 \neq 0$, $\dot{a} = \infty$, $b = 0$, and $e = \infty$. The asymptotics near the value $\xi = 0$ $(\tau = \tau_0)$ has the same form as that given above for the closed model, and is valid for $\tau > \tau_0$ and $\tau < \tau_0$. As ξ grows from 0 (i.e., as $|\tau - \tau_0|$ increases), a/A increases from the value A_1 to ∞, and b^4 increases from 0 to D^3/A^2.

The second type of solutions corresponds to $\xi \leq \xi_1$. At $\xi = \xi_1$, $a = 0$, $\dot{a} = \infty$, $b = \infty$, and $e = \infty$. Near $\xi = \xi_1$ $(\tau = \tau_0)$,

$$a/A \approx [27(\xi_1^2 + 3)^2/12\xi_1]^{1/3}\, [c(\tau - \tau_0)/A]^{1/3}.$$

SovietRxiv

As $|\xi|$ increases from the value $|\xi_1|$ (i.e., as $|\tau - \tau_0|$ increases), a^2 grows from zero, while b^4 decreases, and as $|\xi| \to \infty$,

$$a^2 \to \infty, \qquad b^4 \to D^3/A^2.$$

In this case b nowhere vanishes.

For $A_1 = 0$ we obtain

$$a/A = \xi(9+\xi^2), \qquad c(\tau-\tau_0)/A = \pm\left[(9+\xi^2)^{3/2} - 27\right],$$

$$b^4 = (D^3/A^2)\xi^4(9+\xi^2)^{-2}. \tag{20}$$

At $\xi = 0$, $a = 0$, $\dot{a} = \infty$, $b = 0$, $e = \infty$; as $|\xi| \to \infty$, $a^2 \to \infty$, $b^4 \to D^3/A^2$.

Flat model (3). From equations (9) and (10) we have ($\alpha^2 = 0$)

$$\frac{a\dot{a}^2}{c^2} = \frac{9A}{\xi}, \qquad A = \text{const}; \qquad \frac{a}{A} = 9\xi + A_1, \qquad A_1 = \text{const};$$

$$\frac{c(\tau-\tau_0)}{A} = \pm 3\int^{\xi} [\xi a/A]^{1/2}\, d\xi. \tag{21}$$

For $A_1 > 0$ the expression under the radical in the integral is nonnegative for $\xi \geq 0$ and for $\xi \leq -A_1/9$. Again solutions of two types are possible.

For $\xi \geq 0$, introducing λ according to $\xi = (A_1/9)\operatorname{sh}^2\lambda$, from (21) and (8) we obtain

$$a/A = A_1 \operatorname{ch}^2\lambda, \qquad c(\tau-\tau_0)/A = (A_1^2/144)(\operatorname{sh}4\lambda - 4\lambda),$$

$$b^4 = (D^3/9^6A^2)(a/A)^4 \operatorname{th}^{12}\lambda. \tag{22}$$

At $\tau = \tau_0$ the minimum value $a/A = A_1 \neq 0$ is attained; in this case $\dot{a} = \infty$, $b = 0$, $e = \infty$. Near $\tau = \tau_0$,

$$a/A \approx A_1 + 9(4A_1)^{-1/3}[c(\tau-\tau_0)/A]^{2/3}$$

(as in the closed and open models); as $|\tau| \to \infty$, $a/A \to \infty$, $b^4 \to (a/A)^4(D^3/9^6A^2)$.

For $\xi \leq -A_1/9$, introducing λ according to $\xi = -(A_1/9)\operatorname{ch}^2\lambda$, we obtain

SovietRxiv

$$(a/A)^2 = A_1^2 \operatorname{sh}^4 \lambda, \qquad c(\tau - \tau_0)/A = (A_1^2/144)(\operatorname{sh} 4\lambda - 4\lambda),$$

$$b^4 = (D^3/9^6 A^2)(a/A)^4 \operatorname{cth}^{12} \lambda. \tag{23}$$

At $\tau = \tau_0$ the minimum value $a^2 = 0$ is attained; in this case $\dot{a} = \infty$, $b = \infty$, $e = \infty$. Near $\tau = \tau_0$,

$$a/A \approx (81/4\xi_1)^{1/3}[c(\tau - \tau_0)/A]^{2/3};$$

as $|\tau| \to \infty$, $a^2 \to \infty$, $b^4 \to (a/A)^4(D^3/9^6 A^2)$.

For $A_1 = 0$, (21) becomes the Friedmann solution

$$(a/A)^4 = (9^6 A^2/D^3)b^4 = 324c^2(\tau - \tau_0)^2/A^2,$$

which as $\tau \to \tau_0$ $(e \to \infty)$ differs essentially from (22)–(23).

2. In the case where there is a magnetic field, one must consider the system (3)–(4) with constant $B_1^2 \neq 0$. In this case it is not possible to obtain finite expressions.

Let us consider the plane model for $B_1^2 \neq 0$. Introduce the variables

$$\omega = c^2/(a\dot{a})^2, \qquad \xi = -(\omega^2/a)\, da/d\omega^*. \tag{24}$$

The system (3)–(4) $(\alpha^2 = 0)$ is then reduced to the form

$$d\xi/d\omega = (\xi/3\omega^2)\,[\omega(8B_1^4\xi^2 - 14B_1^2\xi + 6) + \xi(8B_1^2\xi - 3)]; \tag{25}$$

$$b^4 = D^3 a^4(\xi\omega)^3/[B_1^2\xi\omega - (\omega - \xi)]^3. \tag{26}$$

The integral curves of equation (25) for $B_1^2 \neq 0$ are shown in Fig. 1. In the sense of ω, one need consider only values $\omega \geq 0$. According to (2), for nonnegative values of $e = 3p$ having physical meaning, the constant D must be > 0. It follows from (26) that for $D > 0$ the quantity b^4 is negative for values of ω and ξ inside the region bounded by the integral curves $\xi = 0$ and $\xi = \omega/(1 + B_1^2\omega)$, so that the inner part of this region (shaded in Fig. 1) need not be considered. In the remaining part of the half-plane $\omega \geq 0$, equation (26) has singular points $O(\omega = 0, \xi = 0)$ and $A(\omega = 0, \xi = 3/(8B_1^2))$, lying on the ξ-axis, which is an integral curve. The singular points O and A have a structure more complicated than ordinary singular points.

Fig. 1.

SovietRxiv

Fig. 1.

Figure 1: Fig. 1.

The arrows indicate the direction of increase of a^2 along an integral curve. The point O corresponds to the minimum of a^2. When moving along an integral curve in the direction of increasing a^2, one must pass from the lower half-plane to the upper one through an infinitely distant point $\omega = \text{const}$, $\xi = \mp\infty$, corresponding to a finite value of a^2. All curves arrive at the point A, at which $a^2 = \infty$, $|\tau| = \infty$.

Near the point O (in the region $\xi < 0$) $\xi \approx -k_1^2\omega^2$, $k_1^2 = \text{const}$, $a \approx \text{const} \cdot \exp(k_1^2\omega)$; at the point O, $a \neq 0$, $b = 0$, $e = \infty$. It is essential that the minimum value of a^2 attained at $e = \infty$ is not equal to 0, and the energy density of the magnetic field $W = B_1^2(c^4/8\pi k)/a^4$ is finite. The asymptotics near the singularity $e = \infty$ in the presence of a magnetic field is expressed by formulas (22) for $\lambda \to 0$ **.

Let us note that for $e = 3p$ (in contrast to the case of dustlike matter) $a^2(\tau)$ and $b^2(\tau)$ are symmetric functions of τ (with an appropriate choice of the time origin).

In the closed and open models, in the presence of a magnetic field, a^2 likewise does not go to 0, since for small a in (3) and (4) the term B_1^2/a^4 dominates over $\pm\alpha^2/a^2$. The asymptotics near $e = \infty$ in the closed and open models is also expressed by formulas (22) for $\lambda \to 0$.

Scientific Research Institute of Mechanics
Moscow State University
named after M. V. Lomonosov

Received
27 XII 1966

CITED LITERATURE

1. I. S. Shikin, DAN, **171**, 73 (1966).

2. A. S. Kompaneets, A. S. Chernov, ZhETF, **47**, 1939 (1964).

3. A. G. Doroshkevich, Astrofizika, **1**, 255 (1965).

*

$\omega \equiv \text{const}$ corresponds to the isotropic Friedmann solution; for $B_1^2 \neq 0$ the system (3)–(4) has no such solution.

**

SovietRxiv

For $B_1^2 = 0$ the pattern of the integral curves

$$\xi = (9A)^2\omega^2/[(9A)^2\omega - 1]$$

differs substantially from Fig. 1. For ($\omega = 1/(9A)^2$, $\xi = \mp\infty$), $a^2 = \infty$, $|\tau| = \infty$. The curves with $\xi < 0$ correspond to (22), the curves with $\xi > 0$ to (23).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.