NON-ISOMORPHISM OF BANACH SPACES OF ANALYTIC FUNCTIONS OF ONE AND OF TWO VARIABLES

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Abstract

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NON-ISOMORPHISM OF BANACH SPACES OF ANALYTIC FUNCTIONS OF ONE AND OF TWO VARIABLES

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I. In analysis, the idea has become widespread that one or another space of functions of a larger number of variables is, in some sense, "more massive" than the corresponding space of functions of a smaller number of variables. However, of the confirmations of this idea one finds in proofs of the non-isomorphism* of certain "natural" spaces of functions of different numbers of variables. Therefore a result obtained by A. A. Milutin in 1951 ([1]) was very unexpected; from it it follows that the spaces of all continuous functions on cubes of different dimensions are isomorphic to one another.

One may try to prove the non-isomorphism of functional spaces by finding their so-called linear dimension**. Denote by D^n the polycylindrical domain of the space of n complex variables, given by the inequalities $|z_i| < 1$ $(i = 1, 2, \dots, n)$. Denote by $H(D^n)$ the space of all functions analytic in the domain D^n, with the topology of uniform convergence on compact subsets of the domain D^n. A. N. Kolmogorov ([2]) showed that the spaces $H(D^n)$ and $H(D^m)$ have different linear dimension if $n \neq m$, and therefore are non-isomorphic.

Let I^k be the k-dimensional unit cube in k-dimensional Euclidean space. Denote by $C^s(I^k)$ the space of all functions, s times continuously differentiable, defined on the cube I^k, with norm

$$\|f(x)\|_{C^s(I^k)} = \sum_{i_1+\dots+i_k \le s} \sup_{x \in I^k} \left| \frac{\partial^{i_1+\dots+i_k} f(x)}{\partial x_1^{i_1} \partial x_2^{i_2} \cdots \partial x_k^{i_k}} \right|.$$

Denote by $A(D^n)$ the space of all those analytic functions in the polycylinder D^n which can be continuously extended to the closed polycylinder $\overline{D}^n$, with norm

$$\|f(z)\|_{A(D^n)} = \sup_{z \in D^n} |f(z)|.$$

From the results of A. L. Pelczyński ([3,4]) it follows that, for any natural s, k, and n,

$$\dim_l C^s(I^k) = \dim_l A(D^n) = \dim_l C(I),$$

where $C(I)$ is the space of continuous functions on the interval I. Thus, in terms of linear dimension these functional spaces, widespread in analysis, cannot be distinguished.

* Two linear topological spaces are customarily called isomorphic if there exists between them a mutually one-to-one, linear and continuous correspondence (linear homeomorphism).

** Linear dimension is denoted by $\dim_l E$, where E is a linear topological space. It is said that $\dim_l E_1 \leq \dim_l E_2$ if E_1 is isomorphic to a closed linear subspace of the space E_2; $\dim_l E_1 = \dim_l E_2$ if $\dim_l E_1 \leq \dim_l E_2$ and $\dim_l E_2 \leq \dim_l E_1$.

Let us now define a finer invariant $r(E)$ of the linear topological space E. We shall say that $r(E_1) \leq r(E_2)$ if E_1 is isomorphic to such a closed subspace $\widetilde{E}_1$ of the space E_2 that there exists a linear continuous operator $S : \widetilde{E}_1^* \to E_2^*$ extending linear functionals $\varphi \in E_1^*$ to linear functionals $S\varphi \in E_2^*$. We shall write that $r(E_1) = r(E_2)$ if $r(E_1) \leq r(E_2)$ and $r(E_2) \leq r(E_1)$. The invariant $r(E)$ will be called the **growth** of the space E.

A. A. Miljutin has proposed the hypothesis that the spaces $C^s(I^n)$ for different n have different growth and therefore are non-isomorphic. So far only a special case of this hypothesis has been proved, namely, it has been shown ([5]) that $r(C^s(I)) \neq r(C^s(I^k))$ for $k \geq 2$.

The spaces $A(D^n)$, apparently, also have different growth for different n. In this direction the following has been proved.

Theorem. *Let $n \geq 2$. Then $r(A(D)) \neq r(A(D^n))$, and consequently the spaces $A(D)$ and $A(D^n)$ are non-isomorphic.*

In the present note a scheme of proof of this theorem is given. It is interesting to recall here one result of A. L. Pelczyński ([4]), which in terms of growth can be formulated in the following form: $r(A(D)) \neq r(C(X))$ for any compact Hausdorff space X, where $C(X)$ denotes the space of all continuous functions on X.

II. Let E be a Banach space, and let E_1^* be the unit ball of the conjugate space, endowed with the weak topology. It is known that then E_1^* is a compact Hausdorff space, and an operator $I_E : E \to C(E_1^*)$ of an isometric embedding of the space E into the space $C(E_1^*)$ is defined by the formula $I_E x = x(x_1^*)$, where $x \in E$, $x_1^* \in E_1^*$, and $x(x_1^*)$ is the value of the functional x_1^* on the element x. Following A. A. Miljutin, we introduce the following characteristic of subspaces M^* of the space E^*.

Denote by $\chi(M^*)$ the lower bound of the norms of the linear extension operators $S : M^* \to C^*(E_1^*)$ of linear functionals from M^* to linear functionals from $C^*(E_1^*)$. In other words, the lower bound is taken over the norms of such linear operators S for which the operator $I_E^* S$ is the identity on the subspace M^*. Here I_E^* denotes the operator conjugate to the operator I_E. In the case when no extension S of bounded norm exists, we shall write that $\chi(M^*) = \infty$.

Lemma 1 (A. A. Miljutin). *Let $J : E \to C$ be some isomorphic embedding of the space E into the space of continuous functions C on a certain compact Hausdorff space. Let $M^* \subset E^*$. Denote by $\chi_J(M^*)$ the lower bound of the norms of the linear operators $S : M^* \to C^*$ such that the operator $J^* S$ is the identity operator on the subspace M^*. Then*

$$\frac{1}{\|J\|}\chi(M^*) \le \chi_J(M^*) \le \|J^{-1}\|\chi(M^*).$$

Here $\|J\|$ denotes the norm of the operator J, and $\|J^{-1}\|$ the norm of the operator inverse to J and defined on the image of the space E under the mapping J.

Let $I_A : A(D) \to C(\partial D)$ be the operator of the natural isometric embedding of the space $A(D)$ into the space of continuous functions on the boundary ∂D of the disk D. The operator I_A assigns to each function $f(z) \in A(D)$ the continuous function formed by the boundary values of the function $f(z)$ on the circle ∂D. We decompose the subspace $C^*(\partial D)$ into the direct sum of subspaces $C^*(\partial D) = L_1 \oplus M_1$, where L_1 is the subspace of functionals given by absolutely continuous measures on ∂D, and M_1 is the subspace of functionals given by singular and discrete measures on ∂D.

Lemma 2. *The space $A^*(D)$ decomposes into the direct sum of subspaces $A^*(D) = L \oplus M$, where $L = I_A^* L_1$, $M = I_A^* M_1$, and $\chi(M) = 1$, while L is separable.*

Let $f(z) \in A(D)$. Then

$$f(z) = \sum_{k=0}^{\infty} a_k(f) z^k, \quad \text{if } |z| < 1.$$

The coefficients $\{a_k(f)\}$ are linear continuous functionals, i.e. $a_k \in A^*(D)$ for all k. Since

$$a_k(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta}) e^{-ik\theta}\, d\theta,$$

we have $a_k(f) \in L$, where $L \subset A^*(D)$ is the subspace of functionals defined in Lemma 2.

Consider the subspace $L_n \subset L$ spanned by the functionals $a_0(f), a_1(f), \dots, a_{n-1}(f)$. Define a linear extension operator $S_n : L_n \to C^*(\partial D)$ from functionals in L_n to functionals in $C^*(\partial D)$ by the formula

$$S_n\left(\sum_{k=0}^{n-1} \lambda_k a_k\right)(f) = \sum_{k=0}^{n-1} \lambda_k \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-ik\theta}\, d\theta, \quad \text{where } f \in C(\partial D).$$

It turns out that the extension operator S is almost the best possible among all operators $S : L_n \to C^*(\partial D)$ such that the operator $I_A^* S$ is the identity on L_n; namely, the following holds.

Lemma 3. $\|S_n\| \le \chi(L_n) + 2$.

Lemma 4. There exist two absolute positive constants γ_1 and γ_2 such that

$$\gamma_2 \ln n \le \|S_n\| \le \gamma_1 \ln n$$

for every n.

From Lemmas 3 and 4 there follows the important result for us that, for every n,

$$\chi(L_n) \ge \gamma \ln n,$$

where $\gamma > 0$ is an absolute constant.

We shall regard the space $A(D^2)$ as a subspace of the space $C(\partial D^2)$ of continuous functions on the torus $\partial D \times \partial D$.

Lemma 5. Let N^* be a separable subspace of the space $A^*(D^2)$. Then there exists $t_0 \in \partial D$ such that, for every functional $\varphi \in N^*$, there is an extension preserving the norm to a measure $\mu_\varphi \in C^*(\partial D \times \partial D)$ with support outside the circle $(\partial D \times t_0) \subset (\partial D \times \partial D)$. Let us explain that by the circle $(\partial D \times t_0)$ we mean the set of all points $(z_1, z_2) \in (\partial D \times \partial D)$ such that $z_2 = t_0$. The measure μ_φ has support outside this circle in the sense that its restriction to the circle $(\partial D \times t_0)$ is equal to zero.

Lemma 6. Denote by $M_{t_0}^*$ the subspace of functionals of the space $A^*(D^2)$ that can be represented by measures on the torus $(\partial D \times \partial D)$ with support on the circle $(\partial D \times t_0)$.

Let $T : M_{t_0}^* \to A^*(D^2)$ be a linear continuous operator which assigns to each $\varphi \in M_{t_0}^*$ some functional $T\varphi \in A^*(D^2)$ such that the functional $T\varphi - \varphi$ can be represented by a measure with support outside the circle $(\partial D \times t_0)$.

Then

$$\|T\varphi\|_{A^*(D^2)} \ge \|\varphi\|_{A^*(D^2)}.$$

Lemma 7. Let M_n^* be an n-dimensional subspace of the space $M_{t_0}^*$, and let T be an operator of the same kind as in Lemma 6. Then

$$\chi(T(M_n^*)) \geq \chi(M_n^*)/\|T\|.$$

Proof of the theorem. Since, for every $n \geq 2$,

$$r(A(D^2)) \leq r(A(D^n)),$$

it suffices to prove that $r(A(D)) \neq r(A(D^2))$. Suppose the contrary, that is, suppose that there exists an isomorphic embedding

$$J : A(D^2) \to A(D)$$

of the space $A(D^2)$ into $A(D)$, that there exists

linear continuous operator $S : A^*(D^2) \to A^*(D)$ such that J^*S is the identity operator in the space $A^*(D^2)$. By Lemma 2, $A^*(D) = L \oplus M$, where the subspace L is separable, and the subspace M is such that $\chi(M) = 1$. By Lemma 5, there is a $t_0 \in \partial D$ such that for every functional $\varphi \in J^*L$ there is an extension of it to a measure $\mu_\varphi \in C^*(\partial D \times \partial D)$ with support outside the circle $(\partial D \times t_0) \subset \partial D \times \partial D$. Let $f(z_1, z_2) \in A(D^2)$, where $(z_1, z_2) \in D^2$. For $z_2 = t_0$ and $|z_1| < 1$,

$$f(z_1, t_0) = \sum_{k=0}^{\infty} a_k(f, t_0) z_1^k.$$

Denote by M_n^* the n-dimensional subspace of functionals from $A^*(D^2)$ spanned by the functionals $a_0(f, t_0), a_1(f, t_0), \ldots, \ldots, a_{n-1}(f, t_0)$. By Lemmas 4 and 5, for every n, $\chi(M_n^*) \geq \gamma \ln n$, where $\gamma > 0$ is an absolute constant. Let $M_{t_0}^*$ be the subspace of functionals defined in Lemma 6. Consider the operator $T : M_{t_0}^* \to A^*(D^2)$, which to each functional $\varphi \in M_{t_0}^*$ assigns the functional $T\varphi = J^* P_M S\varphi \in A^*(D^2)$, where P_M denotes the projection of the space $A^*(D) = L \oplus M$ onto the direct summand M. Denote by P_L the projection of $A^*(D)$ onto the subspace L.

Then

$$\varphi - T\varphi = J^*(P_M + P_L)S\varphi - J^* P_M S\varphi = J^* P_{LS}\varphi.$$

Since the functional $J^* P_{LS}\varphi \in J^*L$, by construction the functional $T\varphi - \varphi$ can be given by a measure with support outside the circle $(\partial D \times t_0)$. Since $M_n^* \subset M_{t_0}^*$, from Lemma 7 we have

$$\chi(T(M_n^*)) \geq \frac{1}{\|T\|}\chi(M_n^*) \geq \frac{\gamma}{\|T\|}\ln n,$$

where n is arbitrary and $\gamma > 0$ is an absolute constant. It follows that $\chi(T(M_{t_0}^*)) = \infty$. We shall show, on the other hand, that $\chi(T(M_{t_0}^*)) < \infty$.

For this, by Lemma 1 it is enough to verify that, for some isomorphic embedding $J_0 : A(D^2) \to C(\partial A)$, the inequality $\chi_{J_0}(T(M_{t_0}^*)) < \infty$ holds. Let $J_0 = I_{AJ}$, where $I_A : A(D) \to C(\partial D)$ is the isometric embedding operator defined above. By Lemma 6 the operator $T = J^* P_{MS}$ is invertible, and therefore the operator J^* is also invertible; moreover the operator $(J^*)^{-1}$ is defined on the subspace $T(M_{t_0}^*) \subset A^*(D^2)$. Denote by $S_M : M \to C^*(\partial D)$ a linear continuous operator extending the functionals from $M \subset A^*(D)$ to linear functionals from $C^*(\partial D)$. This operator exists by Lemma 2.

Since $(J^*)^{-1}T(M_{t_0}^*) \subset M$, the linear continuous operator

$$S_0 = S_M(J^*)^{-1} : T(M_{t_0}^*) \to C^*(\partial D)$$

is defined, and such that $J_0^* S_0$ is the identity operator on the subspace $T(M_{t_0}^*) \subset A^*(D^2)$. Consequently, $\chi(T(M_{t_0}^*)) < \infty$, and we have obtained the desired contradiction. The theorem is proved.

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Note: Figure translations are in progress. See original paper for figures.