On averaging in systems of integro-differential equations

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Abstract

A system of nonlinear integro-differential equations of the form

$$\frac{dx}{dt} = \varepsilon f(t, x, \int_0^t \varphi(t, s, x(s))\, ds), \tag{1}$$

is considered, where $\varepsilon > 0$ is a small parameter. The system (1) is associated with a system of averaged equations

$$\frac{d\xi}{dt} = \varepsilon f_0(\xi), \tag{2}$$

$$f_0(\xi) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(t, x, \int_0^t \varphi(t, s, x)\, ds)\, dt. \tag{3}$$

(here, the integral under the function f is calculated with respect to the explicitly appearing variable s).

A theorem is proved regarding the proximity of the solutions of equations (1) and (2) on an interval of order ε^{-1}. A corollary derived from the proven theorem is formulated as applied to systems of the form

$$\frac{dx}{dt} = \varepsilon F(t, x) + \varepsilon \int_0^t \Phi(t, s, x(s))\, ds.$$

It is noted that the system of integral equations

$$\varphi(t) = \lambda \int_0^t \Gamma(t, s, \varphi(s))\, ds$$

can be reduced to a system of type (3) through differentiation; therefore, the concept of averaging can be introduced for such systems of integral equations.

Bibliography: 3 items.

Full Text

Preamble

This work, published in 1967 (Vol. III, No. 10), extends the averaging methods for differential equations developed by A. N. Filatov [**?**]. We consider a system of equations where the evolution of the state variable x depends on an integral operator, specifically investigating the behavior of solutions over long time intervals.

Let the initial system be defined as:

$$\frac{dx}{dt} = \epsilon f\left(t, x, \int_0^t \phi(t, s, x) ds\right) = \epsilon F(t, x)$$

where $\epsilon > 0$ is a small parameter. Following the methodology in [**?**], we assume the existence of an average value for the function $F(t, x)$ as $t \to \infty$:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T F(t, x) dt = \lim_{T\to\infty} \frac{1}{T} \int_0^T f\left(t, x, \int_0^t \phi(t, s, x) ds\right) dt = f_0(x)$$

The corresponding averaged system is given by:

$$\frac{d\xi}{dt} = \epsilon f_0(\xi), \quad \xi(0) = x(0)$$

Theorem 1

Assume the following conditions hold for $t \geq 0, s \geq 0$ and x in some domain D: 1. The functions $f(t, x, y)$ and $\phi(t, s, x)$ are bounded and satisfy Lipschitz conditions: - $|f(t, x, y)| \leq M$ - $|f(t, x', y') - f(t, x'', y'')| \leq \lambda|x' - x''| + \rho|y' - y''|$ - $|\phi(t, s, x') - \phi(t, s, x'')| \leq \eta(t, s)|x' - x''|$ 2. The integral of the Lipschitz kernel satisfies $\int_0^t \eta(t, s) ds = \psi_0(t)$, where $\psi_0(t) \leq N$ and $\psi_0(t) \to 0$ as $t \to \infty$. 3. The average $f_0(x)$ exists uniformly with respect to $x \in D$. 4. The solution $\xi(t)$ of the averaged system remains in the domain D for the time interval $0 \leq t \leq L\epsilon^{-1}$.

Under these conditions, for any $\eta > 0$, there exists $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$, the inequality $|x(t) - \xi(t)| < \eta$ holds on the interval $0 \leq t \leq L\epsilon^{-1}$.

Proof Sketch

To prove the theorem, we introduce an auxiliary function $u(t, x)$ defined by the integral of the difference between the original and averaged functions:

$$u(t, x) = \int_0^t \left[f\left(\tau, x, \int_0^\tau \phi(\tau, s, x) ds\right) - f_0(x) \right] d\tau$$

Using a smoothing kernel $\Delta_\alpha(x)$ and the properties of the averaged operator, we can estimate the deviation of the trajectory $x(t)$ from the averaged trajectory

$\xi(t)$. By applying the Gronwall-Bellman inequality to the difference $x(t) - \xi(t) - \epsilon u(t, \xi)$, we obtain bounds that depend on the small parameter ϵ and the properties of the integral operator.

Specifically, the remainder term $R(t)$ in the expansion satisfies:

$$|R(t)| \leq 2\alpha\nu\epsilon + \epsilon^2 \int_0^t \ldots ds$$

As $\epsilon \to 0$, the terms involving the integral operator $\phi(t, s, x)$ vanish due to the condition $\psi_0(t) \to 0$. This ensures that the solutions of the original and averaged systems remain close over the scale $O(\epsilon^{-1})$.

Extension to General Integral Equations

The results can be generalized to systems of the form:

$$\frac{dx}{dt} = \epsilon F(t, x) + \epsilon \int_0^t \Phi(t, s, x(s)) ds$$

where $F(t, x)$ and $\Phi(t, s, x)$ satisfy similar regularity conditions. If the average of the combined operator exists:

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \left[F(t, x) + \int_0^t \Phi(t, s, x) ds \right] dt = f_0(x)$$

then the solution $x(t)$ of the full system is approximated by the solution of the averaged system $\frac{d\xi}{dt} = \epsilon f_0(\xi)$ over the interval $t \in [0, L\epsilon^{-1}]$.

References

1. Filatov, A. N. “On the averaging method in systems of integro-differential equations.” *Doklady Akademii Nauk SSSR*, 165(3), 490-492, 1965.
2. Bogolyubov, N. N., and Mitropolsky, Y. A. *Asymptotic Methods in the Theory of Non-linear Oscillations.* Moscow, Fizmatgiz, 1963.
3. Volosov, V. M. “Averaging in systems of ordinary differential equations.” *Uspekhi Matematicheskikh Nauk*, 17(6), 3-126, 1962.

Note: Figure translations are in progress. See original paper for figures.

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