GROUPS WITH COMPLEMENTED SUBGROUPS

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Abstract

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MATHEMATICS

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GROUPS WITH COMPLEMENTED SUBGROUPS*

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A subgroup F of a group G is called **complemented** if there exists a subgroup H such that $G = F \cdot H$ and $F \cap H = 1$.

In the theory of abstract groups, both finite and infinite groups satisfying the condition of complementability for all subgroups with a given property (invariant, characteristic, etc.) have been studied. In particular, groups in which every subgroup is complementable have been studied in considerable detail. Such groups are called **completely factorizable**.

Completely factorizable finite groups were studied by P. Hall ([1]), whose main result can be formulated as follows: a finite group is completely factorizable if and only if it is isomorphic to a subgroup of the direct product of finite groups whose orders are not divisible by squares of prime numbers.

Infinite completely factorizable groups were considered by N. V. Baeva ([2]) (a detailed exposition is in ([3])). Having clarified in detail the structure of completely factorizable groups, she showed that these groups are isomorphically embeddable in suitably chosen full direct products of finite groups whose orders are not divisible by squares of prime numbers.

The notion of complementability of subgroups in one variant or another is meaningful also for topological groups. This notion, in the form of the splitting of extensions, had occurred earlier as well, mainly in connection with the search for various conditions ensuring such a splitting. Alongside such problems, it is expedient to consider properties of topological groups on which complementability conditions have already been imposed, analogous to complete factorizability of abstract groups. Following this analogy, we introduce the definition:

A topological group G is called **completely factorizable** if, for every closed subgroup F of it, there exists a closed subgroup H such that $G = F \cdot H$ and $F \cap H = 1$. The subgroup H is called a **complement** of the subgroup F.

The present note is devoted to the study of properties of certain classes of topological completely factorizable groups.

First we note three simple properties of such groups:

1. If a topological group G is completely factorizable, then every closed subgroup of it is completely factorizable. If K is a compact or open invariant subgroup of it, then the factor group G/K is completely factorizable.
2. A locally compact abelian group is completely factorizable if and only if it is totally disconnected and decomposes into a direct product of cyclic subgroups of prime orders with a distinguished open compact subgroup (the definition of the direct product of groups with a distinguished subgroup was introduced by N. Ya. Vilenkin ([4])).
3. A connected completely factorizable Lie group is trivial, i.e., consists of a single identity element.

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* In the present note the terminology of N. Bourbaki' s book ([5]) is used.

Using known results on discrete completely factorizable groups, we shall prove the following theorem:

Theorem 1. *Every locally compact completely factorizable topological group G is totally disconnected and metabelian.*

Proof. Let G_0 be the connected component of the identity of the group G.

In view of closedness, it is completely factorizable. The group G_0 has an arbitrarily small compact invariant subgroup K, the quotient group G_0/K by which is a connected Lie group. Being completely factorizable, the group G_0/K is trivial, i.e. $G_0 = K$. But since K is arbitrarily small, $G_0 = 1$, i.e. the group G is totally disconnected.

We shall prove the metabelianity of the group G. To this end choose in G four arbitrary elements x_1, x_2, x_3, x_4 and construct the double commutator

$$x = [[x_1, x_2], [x_3, x_4]].$$

Let V be any neighborhood of the element x. There is a neighborhood U of the identity of the group G such that

$$[[Ux_1, Ux_2], [Ux_3, Ux_4]] \subset V.$$

Since G is locally compact and totally disconnected, in the neighborhood U there is an open compact subgroup F. By hypothesis, G is completely factorizable. Therefore for F there is a complement H. Let

$$x_i = f_i \cdot h_i \quad (f_i \in F,\ h_i \in H,\ i = 1, 2, 3, 4).$$

But $F \subseteq U$, and therefore $f_i \in U$. Hence $h_i \in Ux_i$ $(i = 1, 2, 3, 4)$ and

$$[[h_1, h_2], [h_3, h_4]] \in V.$$

Since the intersection $F \cap H$ consists of the identity element alone, the subgroup H is discrete, and therefore metabelian (see ([2])). Hence it follows that

$$[[h_1, h_2], [h_3, h_4]] = 1.$$

Thus, $1 \in V$. Since V was chosen arbitrarily, this means that

$$[[x_1, x_2], [x_3, x_4]] = 1,$$

i.e. the group G is metabelian.

Corollary. *A locally compact completely factorizable group G decomposes into a semidirect product $G = [A]B$ of two completely factorizable Abelian subgroups A and B (the subgroup A enclosed in square brackets denotes an invariant multiplier).*

A description of the structure of completely factorizable locally compact groups is given by another theorem, which is a partial extension of a theorem of N. V. Baeva ([2]) to topological groups:

Theorem 2. *In order that a locally compact group G be completely factorizable, it is necessary, and in the case of compactness of the group also sufficient, that it can be represented in the form of a semidirect product*

$$G = [A]B$$

of two closed completely factorizable Abelian subgroups A and B, of which the invariant subgroup A has at least one decomposition into a direct product of cyclic subgroups of prime orders, each of which is invariant in G, with the indicated open (in A) compact subgroup.

We outline the proof only for the case of a compact group G (the proof of necessity for a locally compact group differs little from that given below for compact G).

Let G be completely factorizable and let $G = [A]B$ be the semidirect product obtained in the corollary to Theorem 1. The subgroup A decomposes into a topological direct product of cyclic subgroups of prime orders. The character group of this group is discrete and decomposes into a direct product of cyclic subgroups of prime orders. The group G induces an Abelian group T of automorphisms of the group A. Each automorphism t from T determines a conjugate automorphism t of the group , for which

$$(t\beta)(a) = \beta(ta), \qquad a \in A,\ \beta \in . \tag{1}$$

The totality of all such automorphisms of the group is algebraically isomorphic to the group T, and therefore we shall denote it also by T. It is easy to see that if H is a T-invariant subgroup of A, then its annihilator $\Psi = (, H)$ is also T-invariant.

The groups A and B have the following property common to them:

(τ) for every closed subgroup there exists a T-invariant complement.

Indeed, if H' is a complement of a closed subgroup F of A in G, then $A = F \cdot (A \cap H')$. But $H = A \cap H'$ is invariant in G, i.e., T-invariant. If Φ is an arbitrary subgroup of B and H is a T-invariant complement of its annihilator $F = (A, \Phi)$, then $\Psi = (B, H)$ is a T-invariant complement of the subgroup Φ in B.

Now choose a decomposition of the group B into prime factors in a special way. It can be decomposed into the direct product $B = B_1' \times \Gamma_1'$, where B_1' is a cyclic subgroup of prime order. In view of property (τ), instead of B_1' one can choose a T-invariant factor B_1 so that $B = B_1 \times \Gamma_1'$, and then replace Γ_1' by a T-invariant factor Γ_1. We obtain $B = B_1 \times \Gamma_1$, where B_1 is a cyclic subgroup of prime order. The subgroup Γ_1 is likewise decomposed into a direct product of cyclic subgroups of prime orders. Repeating for it the argument just given, we obtain a decomposition $\Gamma_1 = B_2 \times \Gamma_2$ into a product of T-invariant subgroups B_2 and Γ_2, where B_2 has prime order. Continuing in this way and applying transfinite induction when necessary, we extract a set of T-invariant cyclic subgroups of prime orders B_μ, $\mu \in M$, such that B decomposes into their direct product

$$B = \prod_{\mu \in M} B_\mu. \tag{2}$$

If now Δ_μ is the subgroup generated by all B_α, $\alpha \neq \mu$, and $A_\mu = (A, \Delta_\mu)$ is its annihilator in A, then the group A decomposes into the topological direct product

$$A = \prod_{\mu \in M} A_\mu \tag{3}$$

of T-invariant cyclic subgroups A_μ of prime orders.

Conversely, suppose that the group G decomposes into the semidirect product $G = [A]B$ of completely factorizable abelian closed subgroups, of which the invariant factor A decomposes into a topological direct product of the form (3), where each factor A_μ is a cyclic subgroup of prime order, invariant in G. Using the preceding notation, we may say that these factors are T-invariant. The character group B of the group A is discrete and decomposes into a direct product of the form (2) with cyclic T-invariant factors of prime orders.

We shall show that the group B has property (τ). Let Φ be an arbitrary subgroup of it. We may assume that the set M of indices μ is well ordered. Extract from M a subset N by the following rule: let α be the first index of the well-ordered set M such that $\Phi \cap B_\alpha = 1$; next, let β be the first index in M such that $(\Phi \times B_\alpha) \cap B_\beta = 1$. Continuing in this way, by means of transfinite induction we obtain a set of indices $N = (\alpha, \beta, ...)$. It can be shown that $\Psi = \prod_{\nu \in N} B_\nu$ is a T-invariant complement of the subgroup Φ. Hence it follows that the group A also has property (τ).

In view of the compactness of the group G, the subgroups A and B are also compact; therefore the mapping $(b, a) \mapsto b \cdot a$ of their Cartesian product (B, A) onto $G = B \cdot A$ is a homeomorphism. It follows that if F is any closed subgroup of G, then the mapping $\varphi : F \to B$, defined by $\varphi(ba) = b$, where $f = ba \in F$, $b \in B$, $a \in A$, is continuous. At the same time it is a homomorphism of F onto some subgroup B_1 of B. Since F is compact, B_1 is also compact and, consequently, closed in B. For sub-

there exists a complement K to the subgroup B_1 in B. The subgroup $A_1 = A \cap F$ in A has a T-invariant complement L. Then L is invariant in G, and the product $H = K \cdot L$ will be a compact subgroup of G. It turns out that H is a complement to the subgroup F. The proof of this assertion is carried out in exactly the same way as for discrete groups.

Corollary. *The topological direct product of an arbitrary set of compact completely factorizable groups is a compact completely factorizable group.*

Theorem 3. *Let S be the topological direct product of an arbitrary set of finite groups whose orders are not divisible by squares of prime numbers. Any closed subgroup G of the group S is completely factorizable. Conversely, every compact completely factorizable group G is isomorphic to a closed subgroup of some topological direct product S of finite groups whose orders are not divisible by squares of prime numbers.*

Proof. The first part of the theorem follows immediately from F. Hall's theorem on finite completely factorizable groups and from the propositions given above. If G is a compact completely factorizable group, then, by Theorem 1, it is totally disconnected and, consequently, is the projective limit of some set of finite completely factorizable groups H_μ, $\mu \in M$. Hence the group G is isomorphically embeddable in their full direct product

$$\prod H_\mu$$

equipped with the Tychonoff topology. By F. Hall' s theorem, each group H_μ is embeddable in a finite group S_μ decomposable into a direct product of finite groups whose orders are not divisible by squares of prime numbers. Therefore the group G is isomorphic to a closed subgroup of the full direct product

$$S = \prod S_\mu$$

with the Tychonoff topology. The group S satisfies the requirements of the theorem.

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Note: Figure translations are in progress. See original paper for figures.