ON ONE METHOD FOR OBTAINING NEW FIXED-POINT PRINCIPLES

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Abstract

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ON ONE METHOD FOR OBTAINING NEW FIXED-POINT PRINCIPLES

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1. Let E and E_1 be two Banach spaces; Ω a certain domain in E; Π its boundary; $\overline{\Omega} = \Omega \cup \Pi$, $F(x, y)$ $(x, y \in \overline{\Omega})$ an operator with values in E_1. Consider the equation

$$F(x, x) = 0. \tag{1}$$

Suppose that the set $\mathfrak{N}$ of solutions of equation (1) is compact, closed, and has no common points with Π.

Assume further that for each point $x_0 \in \mathfrak{N}$ there can be specified such a neighborhood $U(x_0)$ that, for $y \in U(x_0)$, the equation

$$F(x, y) = 0 \tag{2}$$

has in $\overline{\Omega}$ a unique solution $x = R(y)$. It is easy to see that from $y = R(y)$ it follows that y belongs to the set $\mathfrak{N}$. Therefore equation (1) is equivalent to the equation

$$x = R(x). \tag{3}$$

We emphasize that the operator R is already defined not on $\overline{\Omega}$, but only on some neighborhood $U(\mathfrak{N})$ of the set $\mathfrak{N}$, which, generally speaking, is unknown to us. If $\mathfrak{N}$ is empty, then it may happen that the operator R is not defined at a single point.

Finally, let us make the last assumption. Suppose that the operator R, on some closed neighborhood $V(\mathfrak{N}) \subset U(\mathfrak{N})$ $(\overline{V(\mathfrak{N})} = V(\mathfrak{N}))$ of the set $\mathfrak{N}$, is completely continuous. Then the vector field

$$\Phi x = x - R(x) \quad (x \in V(\mathfrak{N})) \tag{4}$$

will be completely continuous; it does not vanish on the boundary Γ of the neighborhood $V(\mathfrak{N})$. Therefore the rotation $\gamma(\Phi;\Gamma)$ of the field (4) on Γ is defined. This rotation does not depend on the choice of the neighborhood of the set $\mathfrak{N}$; hence we shall denote it by $\gamma(\mathfrak{N})$.

By the index μ of equation (1) in the domain Ω we shall call the number $\gamma(\mathfrak{N})$, if it is defined, and zero if $\mathfrak{N}$ is empty. Of course, the index may also turn out to be equal to zero in the case when $\mathfrak{N}$ is nonempty.

We note that one and the same equation $f(x) = 0$ may be written in the form (1) with different operators $F(x, y)$. If different $F(x, y)$ are considered, then we shall arrive at different equations (3). Simple examples show that the number $\gamma(\mathfrak{N})$ depends on the equation (3) to which the transition is made; we shall not dwell here on the character of this dependence.

2. Let us now consider the equation

$$F(x, x, \lambda) = 0 \tag{5}$$

with an operator $F(x, y, \lambda)$ $(x, y \in \overline{\Omega})$, depending on a scalar parameter

$\lambda \in [0, 1]$. We shall assume that $F(x, y, \lambda)$ satisfies a number of requirements.

1°. For each λ the index $\mu(\lambda)$ of equation (5) is defined.

In particular, this requirement means that all equations (5) have no solutions on Π.

2°. If $F(x_n, x_n, \lambda_n) = 0$ and $\|x_n - x^*\| \to 0$, $\lambda_n \to \lambda^*$, then x^* is a solution of equation (5) for $\lambda = \lambda^*$.

Requirement 2° is obviously fulfilled if the operator $F(x, y, \lambda)$ is continuous jointly in the variables.

3°. The set $\mathfrak{M}$ of all solutions of equations (5) is compact.

From requirements 2° and 3° it follows immediately that the set of those values λ for which equations (5) have no solutions is open. Suppose that, for some $\lambda = \lambda_0$, the set $\mathfrak{M}(\lambda_0)$ of solutions of equation (5) is nonempty; then from 2° and 3° it follows that for every neighborhood W of the set $\mathfrak{M}(\lambda_0)$ one can indicate an $\varepsilon > 0$ such that all solutions of equations (5) for $|\lambda - \lambda_0| < \varepsilon$ lie in W.

4°. If $\mathfrak{M}(\lambda_0)$ is nonempty, then one can indicate such an $\varepsilon_0 > 0$ and such a closed neighborhood W_0 of the set $\mathfrak{M}(\lambda_0)$ that the equations

$$F(x, y, \lambda) = 0 \tag{6}$$

have in $\overline{\Omega}$, for $y \in W_0$, $|\lambda - \lambda_0| < \varepsilon_0$, a unique solution

$$x = R(y, \lambda), \tag{7}$$

and the vector fields

$$\Phi(\lambda)x = x - R(x, \lambda) \tag{8}$$

depend continuously on λ, so that on the boundary Γ_0 of the neighborhood W_0 they are homotopic.

From 4° it follows at once that for every λ_0 for which $\mathfrak{M}(\lambda_0)$ is nonempty, one can indicate a neighborhood $|\lambda - \lambda_0| < \varepsilon_0$ in which the index $\mu(\lambda)$ does not change.

Thus, the following is true.

Theorem 1. *Let $F(x, y, \lambda)$ satisfy requirements 1°-4°. Then $\mu(0) = \mu(1)$.*

Theorem 1 is, of course, quite simple. Its difference from the usual assertions ([1], [2]) on the preservation of rotation under a homotopic change of a vector field consists in the dependence on λ of the domain on which the vector field is considered. What is unexpected is that the simple consideration contained in Theorem 1 leads to interesting new fixed-point principles.

3. **Theorem 2.** *Let the operator A map into itself the ball T ($\|x\| \le \rho$) of a Banach space E, and let A be representable in the form*

$$A = B + C, \tag{9}$$

where B is completely continuous and C is a contraction:

$$\|Cx - Cy\| \le q\|x - y\| \qquad (q < 1;\ x, y \in T). \tag{10}$$

Then A has at least one fixed point in the ball T.

Proof. Suppose that A has no fixed points. Put

$$F(x, y, \lambda) = x - \lambda By - \lambda Cx \qquad (x, y \in T;\ 0 \le \lambda \le 1). \tag{11}$$

The operator (11), obviously, satisfies conditions 1°-4°. For $\lambda = 0$ equation (5) has the form $x = 0$, and its index is equal to 1. From Theorem 1 it then follows that $\mu(1) = 1$. This means that the equation

$$x - Bx - Cx = 0 \tag{12}$$

has at least one solution, i.e., A has a fixed point. We have arrived at a contradiction. The theorem is proved.

An assertion close to Theorem 2 (but substantially weaker) was proved in (3). For the case of a Hilbert space, Theorem 2 was communicated to us by R. L. Frum-Ketkov, a conversation with whom in August 1966 served as the impetus for carrying out the present work. As A. S. Schwartz informed us, R. L. Frum-Ketkov has recently proved Theorem 2 and a number of related assertions also for operators in Banach spaces; these results were obtained by him as a result of constructing a theory of the degree of a mapping (modulo two) for new classes of mappings in Banach spaces.

Let us note that Theorem 2 remains valid if the assumption on the invariance of the ball T is replaced by the assumption that from $Ax = \lambda x$ ($\|x\| = \rho$) there follows the inequality $\lambda \leqslant 1$. The proof is not changed. In particular, the following is true.

Theorem 3. *Let an operator A, acting in a Hilbert space on the ball T ($\|x\| \leqslant \rho$), admit the representation (9), and let*

$$(Ax, x) \leqslant (x, x) \qquad (\|x\| = \rho). \tag{13}$$

Then A has in the ball T at least one fixed point.

4. We do not give here other consequences of Theorem 1. It is clear how to obtain the corresponding formulations here if one considers an equation of the form

$$Bx + Dx = 0 \tag{14}$$

with a locally invertible operator D and a completely continuous operator B. Typical requirements on the operator D may be, for example, its monotonicity (in the sense in which it is used in works (4–9), etc.); the theory of concave operators (10), and so on, may be used.

Let us also note that Theorem 1 naturally extends to the cases where the vector fields (4) are not completely continuous but belong to such classes for which rotation is defined (for example, when the fields (4) are weakly continuous and rotation is defined in the sense of Yu. G. Borisovich).

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REFERENCES

1. J. Leray, J. Schauder, UMN, 1, no. 3–4 (1946).

2. M. A. Krasnosel' skii, *Topological Methods in the Theory of Nonlinear Integral Equations*, 1956.
3. M. A. Krasnosel' skii, UMN, 10, no. 1 (63), 123 (1955).
4. H. Schaefer, Jahresber. Deutsch. Math. Verein., 59, 132 (1957).
5. M. A. Krasnosel' skii, Tr. seminara po funktsional' nomu analizu, no. 6, 87 (1958).
6. M. A. Krasnosel' skii, DAN, 126, no. 1, 15 (1959).
7. R. I. Kachurovskii, UMN, 15, no. 4, 213 (1960).
8. G. G. Minty, Proc. Math. Acad. Sci. U. S., 50, no. 6, 1038 (1963).
9. C. L. Dolph, G. G. Minty, On Nonlinear Integral Equations of the Hammerstein Type, *Nonlinear Integral Equations*, Madison, 1964, p. 98–154.
10. M. A. Krasnosel' skii, *Positive Solutions of Operator Equations*, Moscow, 1963.

Note: Figure translations are in progress. See original paper for figures.

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