THE CARTAN–EILENBERG HOMOLOGY THEORY FOR A GENERALIZATION OF THE CLASS OF LIE ALGEBRAS

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Abstract

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THE CARTAN–EILENBERG HOMOLOGY THEORY FOR A GENERALIZATION OF THE CLASS OF LIE ALGEBRAS

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In the note (1) the concept of a D-algebra was introduced, generalizing the concept of a Lie algebra. We define, for D-algebras, the concept of a representation, in connection with which it becomes possible to construct for D-algebras a homology theory by the Cartan–Eilenberg method ((2), Chap. XIII). On the other hand, the theory of extensions, which in the case of Lie algebras leads to the coincidence $\Sigma(\mathcal{L}, \mathfrak{M}) = H^2(\mathcal{L}, \mathfrak{M})$ (see (2), Chap. XIV), for D-algebras differs substantially, in its results, from the theory based on the study of the corresponding cohomology group. For D-algebras the latter corresponds to a proper part of the set of all extensions. By an extension we shall everywhere in this paper mean an extension with abelian kernel. In general, one may call a homology theory for a certain class of algebras classical if for it there holds an analogue of the equality $\Sigma(\mathcal{L}, \mathfrak{M}) = H^2(\mathcal{L}, \mathfrak{M})$. Thus, the main assertion of the present paper is that the Cartan–Eilenberg homology theory for D-algebras is not classical.

We shall consider here only left D-algebras; the symbol $\mathfrak{Z}$ in a left D-algebra $\mathcal{D}$ over a field K (of characteristic zero) will denote the minimal L-ideal in $\mathcal{D}$, i.e. the least of those ideals in $\mathcal{D}$ whose quotient algebras are Lie algebras. The definition of associativity of a distributive algebra $\mathcal{D}$ and of an associative algebra $\mathfrak{A}$, used in the present paper, differs from that introduced in (1) in that condition 1) is replaced by the following condition: the vector space of the algebra $\mathcal{D}$ is a subspace of $\mathfrak{A}$. It can be proved that the multiplication law in a distributive algebra polynomially associated with $\mathfrak{A}$ is determined by the formula $a * b = \xi_1 ab + \xi_2 ba$, where $\xi_i \in K$ are certain fixed constants.

Theorem 1. *Let $\mathfrak{A}$ be an associative algebra, and let $\mathcal{D}$ be a D-algebra polynomially associated with it. If $\mathcal{D}$ contains at least one element a for which $a^3 \neq 0$, then $\mathcal{D}$ is a Lie algebra and $a * b = \xi(ab - ba)$, $\xi \in K$ a fixed constant.*

Proof. In a (left) D-algebra the equality $(a * a) * a = 0$ must hold; passing to the above-indicated form of the multiplication formula in $\mathcal{D}$, we obtain the assertion of the theorem.

This theorem contains Theorem 2 of (^{1}). In what follows it will be assumed that the condition of Theorem 1 is satisfied and that $\xi = 1$. Let us consider the application of this theorem to the representation theory of D-algebras. Let $\mathfrak{M}$ be a vector space over K; let $\mathcal{D}$ be a D-algebra. A mapping $Q : \mathcal{D} \to \mathcal{E}(\mathfrak{M})$, where $\mathcal{E}(\mathfrak{M})$ denotes the algebra of linear transformations of $\mathfrak{M}$, is called a **representation** of $\mathcal{D}$ if there exists a bilinear function $f : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ such that for all $a, b \in \mathcal{D}$ the equality

$$Q(a * b) = f(Q(a), Q(b))$$

holds. A representation is called **polynomial** if $f(x, y)$ is a polynomial in the (noncommuting) unknowns x, y over K. We shall also say that by means of the representation Q a D-module $\mathfrak{M}$ over $\mathcal{D}$ is given.

It follows from Theorem 1 that if $Q : \mathcal{D} \to \mathcal{E}(\mathfrak{M})$ is a D-module such that $Q(a)^3 \neq 0$ for at least one $a \in \mathcal{D}$, then $Q(\mathcal{D})$ is a Lie algebra. Thus Q narrows to a homomorphism $Q : \mathcal{D} \to \mathcal{E}_L(\mathfrak{M})$, where $\mathcal{E}_L(\mathfrak{M})$ ···

denotes the Lie algebra adjoined to $\mathcal{E}(\mathfrak{M})$. A homomorphism $Q : \mathcal{D} \to \mathcal{E}_L(\mathfrak{M})$ will be called an L-representation of the D-algebra $\mathcal{D}$. It is obvious that the kernel of every L-representation contains $\mathfrak{Z}$. In what follows only polynomial L-representations will be considered. Under this assumption, every D-module $\mathfrak{M}$ may be regarded as an L-module over the Lie algebra $\mathcal{L} = \mathcal{D}/\mathfrak{Z}$.

Definition. The **Cartan-Eilenberg homology groups** of a D-algebra $\mathcal{D}$ with coefficients in a D-module $\mathfrak{M}$ are the homology groups of the Lie algebra $\mathcal{L} = \mathcal{D}/\mathfrak{Z}$ with coefficients in $\mathfrak{M}$, regarded as an $\mathcal{L}$-module.

Let us now consider the relation of the homology theory thus defined to the theory of abelian extensions in the class of D-algebras. Let $\mathfrak{F}$ be a vector space; $\overline{\mathcal{D}}$ a D-algebra, and $0 \to \mathfrak{F} \to \mathcal{D} \to \overline{\mathcal{D}} \to 0$ an extension of $\overline{\mathcal{D}}$ in the class of D-algebras. The mapping $P : \overline{\mathcal{D}} \to \mathcal{E}_L(\mathfrak{F})$, defined by the formula $P(\bar{l})f = l * f$, endows $\mathfrak{F}$ with the structure of a (left) D-module. Identifying now the vector spaces $\mathcal{D} = \mathfrak{F} + \overline{\mathcal{D}}$, the law of multiplication in $\mathcal{D}$ (denoted by $*$) may be expressed in the following way:

$$f_1 * f_2 = 0; \qquad l_1 * f_1 = P(l_1) f_1; \qquad f_1 * l_1 = -P(l_1) f_1 + z(f_1, l_1);$$

$$l_1 * l_2 = l_1 \overline{*} l_2 + g(l_1, l_2),$$

where $f_i \in \mathfrak{F}$, $\underline{l_i} \in \overline{\mathcal{D}}$, and $z(f, l)$, $g(l_1, l_2)$ are bilinear functions, defined respectively on $\mathfrak{F} \times \overline{\mathcal{D}}$, $\overline{\mathcal{D}} \times \overline{\mathcal{D}}$, with values in $\mathfrak{F}$; $\overline{*}$ is the symbol for multiplication in $\overline{\mathcal{D}}$. It can be shown that the fulfillment of the system of relations

$$z(f, l_1) * l_2 = 0;$$

$$l_1 * (f * l_2) = (l_1 * f) * l_2 + f * (l_1 * l_2);$$

$$l_1 * (l_2 * l_3) = (l_1 * l_2) * l_3 + l_2 * (l_1 * l_3)$$

for all $f \in \mathfrak{F}$, $l_i \in \overline{\mathcal{D}}$ is the condition necessary and sufficient for $\mathcal{D}$ to be a D-algebra and an extension of $\overline{\mathcal{D}}$ with abelian kernel $\mathfrak{F}$ relative to the representation P. On the other hand, equivalent extensions may differ only by the different manner of representing $\mathcal{D}$ as the direct sum of $\mathfrak{F}$ and $\overline{\mathcal{D}}$. Passing from these conditions to conditions on the system of functions $\{z(f,l); g(l_1,l_2)\}$ (the system of factors), we obtain the theorem:

Theorem 2. In order that the D-algebra $\mathcal{D}$ be an abelian extension of the $\overline{D}$-algebra $\overline{\mathcal{D}}$ relative to the L-representation $P : \overline{\mathcal{D}} \to \mathcal{E}_L(\mathfrak{F})$ with system of factors $\{z, g\}$, it is necessary and sufficient that

$$P(l_2)z(f,l_1) = z(z(f,l_1),l_2);$$

$$P(l_1)z(f,l_2) = z(P(l_1)f,l_2) + z(f,l_1 \overline{*} l_2);$$

$$g(l_1, l_2 \overline{*} l_3) - P(l_3)g(l_1,l_2) + z(g(l_1,l_2),l_3) =$$

$$= g(l_1, l_2 \overline{*} l_3) - g(l_2, l_1 \overline{*} l_3) + P(l_1)g(l_2,l_3) - P(l_2)g(l_1,l_3)$$

for all $f \in \mathfrak{F}$, $l_i \in \overline{\mathcal{D}}$. Moreover, two such extensions are equivalent if and only if there exists a linear function $h : \overline{\mathcal{D}} \to \mathfrak{F}$ such that the systems of factors $\{z, g\}$, $\{z', g'\}$ are related by

$$z'(f,l) = z(f,l);$$

$$g'(l_1,l_2) = g(l_1,l_2) - h(l_1 \overline{*} l_2) + P(l_1)h(l_2) - P(l_2)h(l_1) + z(h(l_1),l_2)$$

for all $f \in \mathfrak{F}$; $l, l_i \in \overline{\mathcal{D}}$.

It follows from this theorem that the set $S(\overline{\mathcal{D}}, \mathfrak{F})$ of all pairwise inequivalent extensions determined by the D-module $\mathfrak{F}$ over $\overline{\mathcal{D}}$ can be divided into disjoint classes $S(\overline{\mathcal{D}}, \mathfrak{F}; z(f,l))$, each of which contains the extensions having the corresponding factor $z(f,l)$. Each

one of the classes $S(\overline{\mathfrak{D}}, \mathcal{F}; z)$ can in the usual way be turned into an abelian group.

Let $\overline{\mathfrak{Z}}$ denote the minimal L-ideal in $\overline{\mathfrak{D}}$, $\overline{\mathcal{L}} = \overline{\mathfrak{D}}/\overline{\mathfrak{Z}}$, $\overline{P}$ the narrowing of P to $\overline{\mathcal{L}}$. We note that the group of extensions $\Sigma(\overline{\mathcal{L}}, \mathcal{F})$, defined in the class of Lie algebras over the L-module $\mathcal{F}$, is a subgroup in $S(\overline{\mathfrak{D}}, \mathcal{F}; 0)$. Indeed, if $\overline{A} \in \Sigma(\overline{\mathcal{L}}, \mathcal{F})$ is an extension defined by a system of factors $\{\overline{g}(\bar{l}_1, \bar{l}_2)\}$, $\bar{l}_i \in \overline{\mathcal{L}}$, then one may set $A \in S(\overline{\mathfrak{D}}, \mathcal{F}; 0)$—the extension defined by the system of factors $\{g(l_1, l_2)\}$, $l_i \in \overline{\mathfrak{D}}$, where $g(l_1, l_2) = \overline{g}(\bar{l}_1, \bar{l}_2)$ for all $l_i \in \bar{l}_i$.

However, as follows from example 1 in (^{1}), for every D-algebra $\mathfrak{D}$ and every one of its D-modules $\mathcal{F}$ there exist nonzero extensions not contained in $S(\overline{\mathfrak{D}}, \mathcal{F}; 0)$. This completes the proof of the following theorem:

Theorem 3. *The Cartan-Eilenberg homology theory for the class of left D-algebras is not classical.*

We do not know whether a classical homology theory exists for D-algebras.

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REFERENCES

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^{2} H. Cartan, S. Eilenberg, *Homological Algebra*, IL, 1960.

Note: Figure translations are in progress. See original paper for figures.

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