THE MAXIMUM PRINCIPLE

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Abstract

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THE MAXIMUM PRINCIPLE

1. In the present communication a general method is indicated for deriving theorems of maximum-principle type, generalizing, in particular, the results of ([2–5]). It is based on considerations used in ([1]). As in ([1]), we consider functions $u(x)$ in an n-dimensional domain G, bounded and lower semicontinuous, and possessing an absolutely continuous support mapping. Suppose that such a function satisfies, at least at almost all of its points of convexity, the inequality

$$F(u_{ij}, u_i, u, x) \leqslant 0$$

where the derivatives u_i, u_{ij} may be understood as coefficients of approximating differentials (they include both ordinary and generalized derivatives and, as shown in ([1]), exist at almost all points of convexity).

Let $E^m = E$ be an m-dimensional plane passing through the origin of coordinates O, $1 \leqslant m \leqslant n$, x_E the projection of the point x, and M_E the projection of the set M. Suppose that at the point x_0, $u_{ij}\, dx^i dx^j \geqslant 0$. The projection f_E of the function $f(x) = u_{ij}(x_0) x^i x^j$, i.e.

$$f_E(x') = \inf_{x_E = x'} f(x), \qquad x' \in E,$$

will be a quadratic form of rectangular coordinates in E. We denote its determinant by $w_E(x_0)$. If $m = n$, then $u_E = u$, $w_E = \det(u_{ij})$.

By a pencil we shall mean a set of planes E^m passing through some given E^{m-1} (a point, if $m = 1$). If $m = n$, then the pencil consists of the single "plane" E^n, and then all further qualifications concerning pencils are omitted.

We introduce the condition:

(F). From the fact that the function u satisfies inequality (1), it follows that for the planes E of some pencil $\{E\}$, at almost all points of convexity of u, at which for some $\varepsilon, \varepsilon' > 0$

$$u(x) < \inf u + \varepsilon, \qquad |\nabla u(x)| < \varepsilon'$$

and ∇u is parallel to E, the inequalities

$$w_E(x) \leqslant X_E(x) U_E(\nabla u). \tag{2}$$

hold.

Here the functions $X_E \geqslant 0$ are defined in G and are such that in a neighborhood of any point $x_0 \in G$ each of them is majorized by a summable function of x_E. (The indices E in X, U signify only that these functions depend on E.) In deriving (2) from (1), substitutions $u = u(x)$, $u_i = u_i(x)$, $u_{ij} = u_{ij}(x)$ are, of course, allowed. The generality of condition (F) is clarified in ([1]); it is satisfied, one may say, for any elliptic F.

Extend u to $G + \partial G$, putting, for $x \in \partial G$,

$$u(x) = \liminf u(x'), \qquad x' \to x,\ x' \in G.$$

Introduce the set

$$N \in \{\, u(x) = \inf_x u,\ x \in G + \partial G \,\}. \tag{3}$$

We shall call a cone a finite solid cone with vertex at O, and shall say that it is a cone around the direction v if it contains a segment issuing from O in the direction v.

Theorem 1. *If, under all the conditions listed, the set N has a supporting plane P not intersecting $N \cap \partial G$, then there exists su-*

cone V about the outer normal ν to P, such that for almost all planes E of the pencil $\{E\}$, under condition (F),

$$\int_{V \cap E} \frac{dy}{U_E(y)} < \infty$$

(U_E is a function of a vector, and here, correspondingly, y is the point that is its endpoint).

The proof is based on two lemmas, in which the conditions of Theorem 1 are meant.

Lemma 1. *The supporting image of any neighborhood of the set $N \cap P$ contains a cone about ν.*

The proof is obvious from the geometric consideration of supporting planes to the graph of the function u in $(n+1)$-dimensional space.

Lemma 2. *For every $M \subset G$ and almost all $E \in \{E\}$,*

$$\int_{\psi(M)\cap E} U_E^{-1}(y)\,dy \le \int_{M_E} Y_E(y)\,dy,$$

where $\psi(M)$ is the supporting image of M, and Y_E is such a function that on M, $X_E(x) \le Y_E(x_E)$.

The proof is contained in the derivations of § 1 of [1]. Combining Lemmas 1 and 2 obviously gives Theorem 1. This theorem also serves as the basis for studying the properties of the set N, and thereby for deriving theorems of maximum-principle type.

Addition to Theorem 1. *If $u(x)$ is differentiable at all points of convexity and twice differentiable at them except, at most, for a countable set of them, and if inequalities (2) are also satisfied, then the assertion of Theorem 1 holds for all $E \in \{E\}$; in particular, the pencil $\{E\}$ may consist of a single plane.*

Accordingly, under these conditions, in the corollaries of Theorem 1 one may mean one given plane E.

2. From Theorem 1 one easily derives the following result (Theorem 3 of [1]), where, as below, a pencil $\{E\}$ of positive measure is meant (in the sense of measure in the complete pencil with the same axis E^{m-1}).

Theorem 2. *If, for almost all $E \in \{E\}$, the functions U_E are such that the integrals of U_E^{-1} over any neighborhood of the origin O in E are infinite, then* $\inf u$ *is attained on ∂G.*

The simplest example is the linear inequality

$$a^{ij}u_{ij} + b^i u_i + cu \le 0, \qquad a^{ij}\xi_i\xi_j \ge 0. \tag{4}$$

It is enough to consider it for $u \le \inf u + \varepsilon$, and therefore the term cu may be discarded if $\inf u \ne 0$ and $c \inf u \ge 0$.

For the given plane E, define the quantity a_E: if by a rotation of axes E is made the plane $(x^1, \dots, x^m)$, then $a_E = \det(a^{ij})$, $ij \le m$. Then, for $d^2u \ge 0$,

$$a^{ij}u_{ij} \ge m(a_E w_E)^{1/m}. \tag{5}$$

Therefore, if $a_E \ne 0$, the term cu is absent and ∇u is parallel to E, then from (4) it follows that

$$w_E \le m^{-m} a_E^{-1} |b_E|^m |\nabla u|^m, \tag{6}$$

where b_E is the projection of the vector $(b^1, \dots, b^n)$ onto E. This is inequality (2) with

$$X = m^{-m} a_E^{-1} |b_E|^m, \quad U = |\nabla u|^m.$$

The integral of $U^{-1}(y) = |y|^{-m}$ over the whole cone in E with vertex $y = 0$ is infinite.

Let $\inf u = 0$ and let it be attained at a point $x' \in G$. If x is a point of convexity of u, then for every x'

$$u(x) - u(x') \leq (x - x')\nabla u(x).$$

Therefore, if $u(x') = 0$ and $\nabla u(x)$ is parallel to E, then

$$u(x) \leq |x_E - x'_E|\,|\nabla u(x)|. \tag{7}$$

Therefore from (4) and (5) we easily obtain:

$$w_E \leq m^{-m} a_E^{-1} \big(|b_E| + |c_-|\,|x_E - x'_E|\big)^m |\nabla u|^m. \tag{8}$$

This is again inequality (2) with $U = |\nabla u|^m$. Applying to it, and also to (6), Theorem 2, we obtain:

Theorem 3. Let $u(x)$ satisfy inequality (4), and suppose that for the planes E of some pencil the functions $a_E^{-1}|b_E|^m$ are locally majorized by summable functions of x_E. Suppose, moreover, that either $\inf u \neq 0$ and $c \inf u \geq 0$, or $\inf u = 0$ and, for every $x' \in G$, in some neighborhood of it the functions

$$a_E^{-1}|c_-|^m |x_E - x'_E|^m$$

are majorized by summable functions of x_E. Then $\inf u$ is attained on ∂G.

3. Let us now assume the following:

(A). All the conditions of item 1 hold locally, i.e. for the restriction of u to any subdomain $D \subset G$.

(B). This property is preserved under any regular mapping (analytic with Jacobian $\neq 0$) of the space onto itself.

Let us note that, with regard to the absolute continuity of the supporting map, this is true, in particular, for functions having generalized derivatives u_{ij}, summable with the n-th power, and also, for example, for differentiable functions having everywhere, except possibly at a countable set of points, finite upper derivatives u_{ii}. As for condition (F), it is obviously true for inequality (4).

Under condition (A) all the preceding conclusions hold for each $D \subset G$, which gives substantially greater information about the set N. In particular, from Theorem 1 it follows immediately: if in condition (F) $m = n$, i.e. (2) reduces to

$$\det(u_{ij}) \leq XU,$$

and if the integral of U^{-1} over any cone is infinite, then the set $M = N \cap G$ has no points of local strict convexity (i.e. no such x for which there exist half-spaces R and neighborhoods W such that $M \cap R \cap W = (x)$).

It follows from this that, if condition (B) is also satisfied, then either M is empty or $M = G$. Indeed, suppose that M is nonempty and $\neq G$. Then there is a point $x \in M$ which is a vertex of some solid paraboloid P containing in a neighborhood of x no other points of M. A mapping that carries P into a half-space makes x a point of local strict convexity of the transformed set M, and, according to the preceding, this is impossible.

We apply this conclusion to inequality (4), since for $m = n$ in (6) and (8) $U = |\nabla u|^n$. Under transformation of inequality (4) the coefficients b^i acquire addends that are linear with respect to a^{ij}. Therefore we obtain the theorem:

Theorem 4. Let $u(x)$, under conditions (A), (B), with regard to the absolute continuity of the supporting map, satisfy inequality (4) with locally summable

$$a^{-1}|b^i|^n, \qquad a^{-1}|a^{ij}|^n,$$

where $a = \det(a^{ij})$. Then, if $\inf u \neq 0$ and $c \inf u \geq 0$, or $\inf u = 0$ and, in a neighborhood of every $x' \in G$, the function

$$a^{-1}|c_-|^n|x - x'|^n$$

is summable, then either the set $M = N \cap g$, where $u = \inf u$, is empty, or $M = G$.

4. The application of similar, more delicate considerations in the general case $m \leq n$ leads to the following result.

Suppose that, for a given m, to almost every $x \in G$ there is assigned a plane $E(x)$ passing through it in such a way that: 1) $E(x)$ is determined by the eigenvectors of the matrix $a^{ij}(x)$ corresponding to positive eigenvalues.

values $\alpha_1, \dots, \alpha_k$, $k \geq m$; 2) to each $x_0 \in G$ there corresponds a neighborhood $W(x_0)$ and a number $a(x_0) > 0$ such that for almost every $x \in W(x_0)$ the product of any m of $\alpha_1, \dots, \alpha_k$ is not less than $a(x_0)$. Then we say that there is a field of m-ellipticity $\mathcal{E}^m$. (According to this definition, the dimensions k of the planes $E(x)$ may be different at different x. In the simplest case $k = m$.)

We shall say that a plane $P(x_0)$ passing through x_0 is tangent at x_0 to the field $\mathcal{E}^m$ if almost all $E(x) \in \mathcal{E}^m$ at points x close to x_0 form with $P(x_0)$ arbitrarily small angles. Finally, let us call a nonempty closed set S a generalized l-integral manifold of the field $\mathcal{E}^m$ if at every point $x \in S$ the l-dimensional plane $P(x)$, tangent to $\mathcal{E}^m$, is contained in the contingence of the set S at the point x.

Theorem 5. *Let a function $u(x)$, with the same property as in Theorem 4, satisfy inequality (4), which has a field of m-ellipticity $\mathcal{E}^m$. Suppose, moreover, that one of the following conditions is fulfilled: (I) $\inf u = 0$, and under every* *regular transformation $(x^i) \to (y^i)$, for each point $x_0 \in G$ there is a neighborhood in which the functions*

$$a^{ij}|m, \ |b^i|m, \ |c\_|m \left[\sum_1^m (y^i - y_0^i)^2\right]^{m/2}$$

are majorized by summable functions of $y^1, \ldots, y^m$ only; (II) $\inf u \neq 0$, $c \inf u \geq 0$, and a^{ij}, b^i satisfy the same condition.

Then the set $M = N \cap G$ is either empty or is a generalized m-integral manifold of the field $\mathcal{E}^m$.

For $m = n$ this gives Theorem 4, since if (almost everywhere) $a \neq 0$, then, dividing (4) by $a^{1/n}$, we obtain inequality (4) with $\det(a^{ij}) = 1$.

The condition on $|a^{ij}|m$, etc., in Theorem 5 looks ineffective. But it is certainly fulfilled for a function $g(x)$ if, in a neighborhood of any point, $|g|$ is estimated by a series $\sum h_i(|x - x_i|)$, convergent for all x except singular points x_i, where $h_i(r) \geq 0$, $r \in (0, \infty)$, are decreasing functions such that the series

$$\sum \int_0^r h_i(r)^{m-1}\, dr$$

converges for every r.

Theorems 4 and 5, in view of the invariance of their conditions with respect to changes of variables, are applicable to functions on analytic manifolds. For example, if a function $u \in W_n^n$ on a compact manifold satisfies a homogeneous linear equation with $a^{ij}\xi_i\xi_j \geq 0$, for which $a^{-1}|a^{ij}|n$, $a^{-1}|b^i|n$ are summable, then $u = \text{const}$ if $c \leq 0$, and either u changes sign or $u = 0$ if $c \geq 0$ and somewhere $c > 0$.

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Note: Figure translations are in progress. See original paper for figures.