A GENERAL EXISTENCE THEOREM FOR CLOSED CONVEX POLYHEDRA

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Abstract

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Corresponding Member of the Academy of Sciences of the USSR A. V. POGORELOV

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In the present note a general existence theorem will be proved for a closed convex polyhedron with faces of prescribed directions and with prescribed values of a monotone function on the faces.

Let α be an arbitrary plane and ω_α a function defined on closed convex polygons lying in planes parallel to the plane α, satisfying the following conditions:

1. The function ω_α is positive and continuous.
2. The function ω_α is invariant under parallel translations. This means that if the polygons P and Q coincide under a parallel translation, then the values of the function ω_α on these polygons are equal.
3. The function ω_α is strictly monotone in the sense that if the polygon Q is a part of the polygon P, then $\omega_\alpha(Q) < \omega_\alpha(P)$.
4. If the polygon P varies in such a way that its area $s(P) \to \infty$, then $\omega_\alpha(P) \to \infty$. If the polygon P degenerates (into a segment), then $\omega_\alpha(P) \to 0$.

Theorem 1. *Let $\alpha_1, \alpha_2, \dots, \alpha_n$ $(n \geq 3)$ be a system of planes not parallel to a single straight line; let $\omega_1, \omega_2, \dots, \omega_n$ be a system of functions defined on polygons parallel to the planes $\alpha_1, \alpha_2, \dots, \alpha_n$, respectively, satisfying conditions 1–4.*

Then, whatever the positive numbers $\varphi_1, \varphi_2, \dots, \varphi_n$ may be, there exists a closed convex polyhedron with $2n$ faces parallel to the planes $\alpha_1, \alpha_2, \dots, \alpha_n$, and with values of the functions ω_k on these faces equal to $\varphi_1, \varphi_2, \dots, \varphi_n$.

This polyhedron has a center of symmetry and is determined uniquely up to a parallel translation.

We begin with the proof of the existence of the center of symmetry and the uniqueness of the polyhedron. Let P be a polyhedron whose existence is asserted by the theorem. Since it has $2n$ faces, and a polyhedron can have no more than two faces of a prescribed direction, it follows that the polyhedron P has exactly two faces for each direction α_k. Construct the polyhedron P^*, symmetric to the polyhedron P with respect to an arbitrary point O. Put the faces of the polyhedra P and P^* into correspondence, considering as corresponding the parallel faces with equally directed outer normals. The values of the functions ω on the corresponding faces of the polyhedra P and P^* are equal; consequently, these faces cannot be fitted one into another by a parallel translation. By a well-known theorem of A. D. Aleksandrov [((1), p. 265)], the polyhedra P and P^* are equal and parallelly situated, i.e., they coincide under a parallel translation. Hence we conclude that the polyhedron P has a center of symmetry. The uniqueness of the polyhedron P is proved analogously.

We now prove the existence of the polyhedron. We shall call an arbitrary system of n positive numbers $\varphi_1, \varphi_2, \dots, \varphi_n$ an abstract polyhedron. If there exists a closed convex polyhedron with $2n$ faces parallel to the planes $\alpha_1, \alpha_2, \dots, \alpha_n$, and with values of the functions ω_k on the faces parallel to the planes α_k equal to φ_k, then we shall call it a realization of the abstract polyhedron $(\varphi_1, \varphi_2, \dots, \varphi_n)$.

Let us now define two manifolds. The elements of the first manifold, which we shall denote by M_1, are closed convex symmetric polyhedra with $2n$ faces parallel to the planes $\alpha_1, \alpha_2, \dots, \alpha_n$. Each such polyhedron is determined by n positive numbers—the values of the support function in the directions perpendicular to the planes α_k. It can be represented by a point of n-dimensional Euclidean space with coordinates equal to the indicated values of the support function. The second manifold, which we shall denote by M_2, consists of abstract polyhedra, and it may be interpreted as the interior of the first coordinate angle of n-dimensional Euclidean space with Cartesian coordinates $\varphi_1, \dots, \varphi_n$.

To each polyhedron from M_1 there naturally corresponds a certain abstract polyhedron from M_2. We assert that this correspondence is a homeomorphism, and consequently every abstract polyhedron admits a geometric realization. The proof of this homeomorphism will be based on the "mapping lemma" of A. D. Aleksandrov (([1]), p. 127), as applied to the manifolds M_1 and M_2. The conditions of this lemma are satisfied. Indeed, the manifolds M_1 and M_2 have the same dimension (n). The manifold M_2 is connected, as a convex set. There exist polyhedra that are certainly realizable. The images of distinct points of M_1 in M_2 are distinct by virtue of the uniqueness proved above (polyhedra that can be made to coincide by a parallel translation are identified). It remains to prove that if some abstract polyhedron is the limit of realizable ones, then it is itself realizable.

Let $P_1, P_2, \dots$ be an infinite sequence of polyhedra from M_1; let $P_1', P_2', \dots$ be the sequence of corresponding abstract polyhedra converging to the abstract polyhedron P. We shall show that the polyhedron P is realizable. Denote by $h_1^k, h_2^k, \dots, h_n^k$ the support numbers of the polyhedron P_k in the directions perpendicular to the planes $\alpha_1, \alpha_2, \dots, \alpha_n$. Without loss of generality, we may assume that each sequence $H_s(h_s^1, h_s^2, h_s^3, \dots)$, $s = 1, \dots, n$, converges to a finite or infinite limit.

We assert that each sequence H_s converges to a finite limit different from zero. Indeed, the surface areas of the polyhedra P_k are bounded in the aggregate.

Therefore, if some sequence H_s converges to an infinite limit, then for sufficiently large k the polyhedron P_k is contained in a cylinder of arbitrarily small radius. In that case, among the faces of the polyhedron there will be one whose diameter is of the order of the diameter of the cylinder, and hence the function ω for this face tends to zero as $k \to \infty$, which is impossible.

If we now suppose that some sequence H_s converges to zero, then for sufficiently large k the polyhedron P_k lies between two arbitrarily close parallel planes. In that case there will be a face which, as $k \to \infty$, degenerates into a segment; that is, again the function ω tends to zero, which is impossible.

Thus all sequences H_s have positive limiting values, and hence it follows that the polyhedra P_k converge to a realization of the limiting polyhedron of the sequence P'_k. Now, on the basis of the above-mentioned lemma of A. D. Aleksandrov, we conclude that the manifolds M_1 and M_2 are homeomorphic, and consequently that all abstract polyhedra are realizable. The theorem is proved.

As a consequence of Theorem 1 we obtain the following.

Theorem 2. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ $(n \geq 3)$ be a system of planes no two of which are parallel to one line; let $f(s, p, v_k)$ be a function of the positive variables s, p and of the unit vectors v_k perpendicular to the planes α_k, satisfying the conditions: 1) the function f is positive, continuous, strictly increasing in the variables s, p, and even in v $(f(s,p,v) = f(s,p,-v))$; 2) as $s \to \infty$, $f(s,p,v) \to \infty$, and as $s \to 0$, $f(s,p,v) \to 0$.

Then, whatever the even positive function $\varphi(\mathbf{v}_k)$ may be, there exists a closed convex polyhedron with $2n$ faces, parallel to the planes a_k, such that the areas s_k, perimeters p_k, and outer normals $\mathbf{v}_k$ of its faces satisfy the conditions

$$f(s_k, p_k, \mathbf{v}_k) = \varphi(\mathbf{v}_k).$$

This polyhedron has a center of symmetry and is determined uniquely up to a parallel translation.

Physical-Technical Institute of Low Temperatures
Academy of Sciences of the USSR

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REFERENCES

1. A. D. Aleksandrov, *Convex Polyhedra*, Moscow–Leningrad, 1950.

Note: Figure translations are in progress. See original paper for figures.