ELEMENTARY PARTICLES IN THE FIELD OF A PLANE ELECTROMAGNETIC WAVE

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Abstract

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The equation for a particle in an electromagnetic field has the form

$$(\hat{\nabla} - i\hat{A} + m)\psi = 0 \tag{1}$$

(the charge is included in A). Here $\nabla = (\nabla_l) = \left(\frac{\partial}{\partial x_l}\right)$; $x_4 = ix_0 = ict$; $A = (A_l)$ is the vector potential of the electromagnetic field; $\hat{A} = A_l\gamma^l$; γ^l are the basic matrices of the equation. Exact solutions of equation (1) have so far been obtained only in a few cases. For particles with a single value of the rest mass (see ([1]))

$$\hat{q}^n(\hat{q}^2 - q^2) = 0, \tag{2}$$

where q is an arbitrary (including operator) 4-vector with mutually commuting components. Denote $H = \hat{\nabla} + m$, $U = -i\hat{A}$, and let $\psi = \psi_0 e^{ipx}$ be the wave function of a free particle ($px = p_{lx}l$, $p^2 + m^2 = 0$). Then $H\psi = (i\hat{p} + m)\psi = 0$, while equation (1) takes the form $(H + U)\psi_1 = 0$. Formally the latter equation may be written in the form $H(1 + H^{-1}U)\psi_1 = 0$. Comparing with $H\psi = 0$ (see (2)), we conclude that $(1 + H^{-1}U)\psi_1 = \psi$, whence

$$\psi_1 = (1 + H^{-1}U)^{-1}\psi = \left[1 - H^{-1}U + (H^{-1}U)^2 - ...\right]\psi. \tag{3}$$

As shown in ([2]), the operator inverse to H, owing to (2), can be represented in the form $H^{-1} = P_1(\hat{\nabla}) + (m^2 - \Box)^{-1}P_2(\hat{\nabla})$, where P_1 and P_2 are certain polynomials. Obviously, $(m^2 - \Box)^{-1}e^{iqx}$, and the action of $(m^2 - \Box)^{-1}$ on a function of a more general form is determined from this by expanding the latter in a series or Fourier integral. Let

$$A = af(\varphi), \qquad \varphi = kx = k_{lx}l, \qquad k^2 = ka = 0, \tag{4}$$

i.e. the electromagnetic field has the character of a plane wave. From (3), (4) it is clear that the expression for ψ_1 will have the form

$$\psi_1 = \chi(\varphi)e^{ipx}, \qquad \chi(\varphi) = \Phi(k, p, \varphi)\psi_0, \tag{5}$$

where $\Phi(k, p, \varphi)$ is a matrix whose elements depend on x_l only through $\varphi = k_{lx}l$. The preceding arguments, despite their formal character, have the significance that they lead to the conclusion that it is possible to seek a solution of equation (1) for the field (4) in the form (5) in the case of a particle with arbitrary spin. Substituting (4), (5), and (1), we obtain

$$\hat{k}\chi' + (i\hat{b} + m)\chi = 0, \qquad \chi' = d\chi/d\varphi, \qquad b = p - af. \tag{6}$$

Thus, in the general case, the problem is reduced to solving a system of ordinary first-order differential equations for the function $\chi = \chi(\varphi)$.

The matrix $\hat{k}$ is singular, since according to (2), (4) $\hat{k}^{n+2} = k^2\hat{k}^n = 0$. Therefore the differential equation (6) contains additional algebraic conditions on the function χ. Their extraction is complicated by the fact that the matrix $\hat{k}$ is not reducible to diagonal form, since its minimal equation $\hat{k}^{n+2} = 0$ has multiple zero roots. This makes it difficult to apply the method of projective operators [(4)]. Therefore we multiply equation (6) by the matrices of the invariant bilinear form η. We obtain the equation

$$\alpha\chi' + \eta(i\hat{b} + m)\chi = 0, \quad (\alpha = \eta\hat{k} = -\alpha^+), \tag{7}$$

which is completely equivalent to equation (6), since $\eta^2 = 1$. However, in (7) the matrix α will be anti-Hermitian [(4)], as a result of which its minimal equation will not contain multiple roots and it can be written in the form $aP_0(a) = 0$, with $P_0(0) \neq 0$. Therefore $P_0(a)/P_0(0)$ will be a projective operator selecting, without exception, all vectors χ for which $a\chi = 0$ [(1)]. Multiplying (7) by $P_0(a)$, we obtain $P_0(a)\eta(i\hat{b}+m)\chi = 0$, whence it follows that $\eta(i\hat{b}+m)\chi = \alpha\chi_1 = \eta\hat{k}\chi_1$, where χ_1 is an arbitrary vector. Hence, in turn, it follows that

$$\chi = (i\hat{b} + m)^{-1}\hat{k}\chi_1. \tag{8}$$

The operator $(i\hat{b} + m)^{-1}$ is not difficult to find with the aid of the minimal equation (2); it has the form (see [(2)])

$$(i\hat{b} + m)^{-1} = \frac{1}{m}\sum_{k=0}^{n-1}\left(-\frac{i}{m}\hat{b}\right)^k + \left(-\frac{i}{m}\right)^n \frac{\hat{b}^n(m - i\hat{b})}{b^2 + m^2}, \tag{9}$$

The function χ_1 in (8), generally speaking, may depend on φ in an arbitrary manner. The simpler case is that in which the dependence on φ is concentrated in a scalar factor $F(\varphi)$ with a constant vector χ_0. Taking (9) into account, in order to get rid of the denominator $b^2+m^2=a^2f^2-2apf$, which depends on φ, it is convenient to include explicitly in F the factor b^2+m^2 and to seek the solution in the form

$$\chi = F(\varphi)(b^2+m^2)(i\hat{b}+m)^{-1}\hat{k}\chi_0. \tag{10}$$

We shall apply the general approach described above to particles with the lowest spins $1/2, 0$, and 1.

1. **Spin** $1/2$**.** In this case the Dirac algebra holds:

$$\hat{p}\hat{q}+\hat{q}\hat{p}=2pq, \quad \hat{q}^2=q^2, \tag{11}$$

$$\hat{k}^2=\hat{k}\hat{a}+\hat{a}\hat{k}=\hat{k}\hat{a}\hat{k}=0, \quad (b^2+m^2)(i\hat{b}+m)^{-1}=m-i\hat{b}.$$

Substituting $\chi=F(m-i\hat{b})\hat{k}\chi_0$ into (6), we obtain

$$\left(F'-\frac{b^2+m^2}{2ikp}F\right)\hat{k}\chi_0=0, \tag{12}$$

whence it follows that

$$F=\exp\left(-\frac{i}{2kp}\int(b^2+m^2)\,d\varphi\right). \tag{13}$$

Thus, the solution of equation (1) has the form (see (5))

$$\psi=(m-i\hat{b})\hat{k}\chi_0\exp i\left[px-\frac{1}{2kp}\int(b^2+m^2)\,d\varphi\right], \tag{14}$$

where χ_0 is an arbitrary bispinor. This solution was obtained in another way and in another form by D. M. Volkov [5].

2. Spins 0 and 1

In this case the Duffin-Kemmer algebra is valid

$$\hat{p}\hat{q}\hat{r}=\hat{r}\hat{p}\hat{q}=pq\cdot r+rp\cdot q, \qquad \hat{q}^3=q^2\hat{q}, \qquad \hat{q}\hat{r}\hat{q}=qr\cdot\hat{q}, \tag{15}$$

$$\hat{k}^3=\hat{k}\hat{a}\hat{k}=\hat{a}\hat{k}\hat{a}=\hat{k}^2\hat{a}+\hat{a}\hat{k}^2=0,$$

$$(b^2+m^2)(i\hat{b}+m)^{-1} = \frac{1}{m}\,[b^2+m^2+i\hat{b}(i\hat{b}-m)]. \tag{16}$$

Substituting (10), (16) into (6), with the same value of F (13), after some transformations, taking (15) into account, we obtain

$$\{F'(i\hat{p}+m)\hat{k}^2(i\hat{p}+m)+(Ff)'[\hat{p},\hat{k}^2\hat{a}]+(Ff^2)'\hat{k}^2\hat{a}^2\}\chi_0 = 0. \tag{17}$$

In order that this relation be satisfied for constant χ_0, it is necessary that

$$(i\hat{p}+m)\chi_0 = 0, \qquad \hat{k}^2\hat{a}^2\chi_0 = 0, \qquad [\hat{p},\hat{k}^2\hat{a}]\chi_0 = 0. \tag{18}$$

The conditions $\hat{k}^2\hat{a}^2\chi_0 = 0$ and $\hat{k}^2\hat{a}\chi_0 = 0$ follow one from the other as a result of multiplication by $\hat{a}$. Therefore the third equation (18) is a consequence of the first two. We shall now consider separately the cases of spin 0 and 1.

a) **Spin 0.** In this case $\hat{k}^2\hat{a} = 0$, and only the condition is imposed on the function χ_0 that it be a solution of the equation for a free particle,

$$(i\hat{p}+m)\chi_0 = 0.$$

b) **Spin 1.** In this case the wave function has the form $\psi = (\psi_k, \psi_{ln})$, where ψ_k is a 4-vector and ψ_{ln} is an antisymmetric tensor. The operator $\hat{p}$ acts on ψ in the following way:

$$\hat{p}\psi = \hat{p}(\psi_k, \psi_{ln}) = (p_l\psi_{kl},\ p_n\psi_l - p_l\psi_n). \tag{19}$$

The first of equations (18) leads to the relations $p_k\psi_k = 0$, $\psi_{ln} = \frac{i}{m}(p_l\psi_n - p_n\psi_l)$, while from the second it follows that $k_n\psi_n = 0$ and $a_nk_l\psi_{nl} = -kp\cdot a_n\psi_n = 0$. Thus, the conditions (18) are satisfied if

$$p\psi_l = k\psi_l = 0, \qquad \psi_l = C\varepsilon_{lmnr}p_{mk_{na}}r, \tag{20}$$

where ε_{lmnr} is the Levi-Civita symbol.

Thus, the exact solution of the problem of a particle in the field of a plane electromagnetic wave can be written in a single general form at once for three different particles with spins $1/2, 0, 1$,

$$\psi = (b^2+m^2)(i\hat{b}+m)^{-1}\hat{f}\chi_0 \exp i\left(px - \int \frac{b^2+m^2}{2kp}\,d\varphi\right), \tag{21}$$

where the function χ_0, independent of φ, is arbitrary in the case of spin $1/2$, for spin 0 is subject to the condition $(i\hat{p}+m)\chi_0 = 0$, and for spin 1 has the form (20).

Let us note that this solution is valid not only in the case of a monochromatic electromagnetic wave, but also in the case of an arbitrary function $f(\varphi)$. Obviously, the method set forth here is in principle applicable also to particles with higher spins, although the absence or complexity of the algebra of the matrices γ_ν^l, naturally, will make the calculations difficult.

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Note: Figure translations are in progress. See original paper for figures.