VIBRATIONS OF THREE-LAYER BEAMS

THEORY OF ELASTICITY

1967

SovietRxiv

Abstract

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UDC 539.3:534.1

THEORY OF ELASTICITY

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VIBRATIONS OF THREE-LAYER BEAMS

(Presented by Academician L. I. Sedov on 6 IV 1966)

§ 1. The problem of flexural vibrations of a three-layer beam, under certain assumptions, reduces ([1,2]) to the solution of a single partial differential equation for the displacement function $\chi(x,t)$

$$(1-\vartheta k\nabla^2)\nabla^2\nabla^2\chi(x,t)+\frac{\Omega l^4}{D}\frac{\partial^2}{\partial t^2}(1-k\nabla^2)\chi(x,t)=0, \qquad (1,1)$$

where $\nabla^2=\partial^2/\partial x^2$; ϑ, k, Ω, D are constants depending on the elastic, geometric, and mass characteristics of the three-layer bar ([1]); $\chi(x,t)$ is a dimensionless displacement function of the longitudinal coordinate x and time t, related to the deflection $w(x,t)$ by the formula

$$w(x,t)=l(1-k\nabla^2)\chi(x,t) \qquad (1,2)$$

(l is some linear quantity, for example the length of the bar).

We shall seek a solution of the linear differential equation (1,1) in the form

$$\chi(x,t)=\chi(x)e^{i\omega t}, \qquad (1,3)$$

where the amplitude $\chi(x)$, in the general case, is a complex function of the real variable x; ω is the vibration frequency. Substituting (1,3) into (1,1), we find

$$(1-\vartheta k\nabla^2)\nabla^2\nabla^2\chi(x)-\omega_*^2(1-k\nabla^2)\chi(x)=0, \qquad (1,4)$$

where $\nabla^2=\dfrac{d^2}{dx^2}$; $\omega_*^2=\dfrac{\Omega l^4}{D}\omega^2$ is the dimensionless frequency.

Represent the particular solution of (1,4) in the form

$$\chi(x)=e^{\alpha x}. \qquad (1,5)$$

Substituting (1,5) into (1,4), and canceling $e^{\alpha x}\neq 0$, we obtain

$$y^3 + a_4 y^2 + a_2 y + a_0 = 0, \tag{1,6}$$

where $y = \alpha^2$; $a_4 = -1/\vartheta k$; $a_2 = \omega_*^2/\vartheta$; $a_0 = -\omega_*^2/\vartheta k$.

It can be shown that the cubic equation (1,6) has three real roots: one negative ($y_1 = -\lambda^2$) and two positive ($y_2 = \mu^2,\ y_3 = \nu^2$), related by

$$\omega_*^2 = \lambda^4 \frac{1+\vartheta k\lambda^2}{1+k\lambda^2}; \qquad y_{2,3} = \frac{1+\vartheta k\lambda^2}{2\vartheta k}\left(1 \mp \sqrt{1 - \frac{4\vartheta k\lambda^2}{(1+\vartheta k\lambda^2)(1+k\lambda^2)}}\right). \tag{1,7}$$

For $4\vartheta k/(1+\vartheta k\lambda^2)(1+k\lambda^2) \ll 1$, the estimates

$$y_2 \geq \frac{\lambda^2}{1+k\lambda^2}, \qquad y_3 \leq \frac{1}{\vartheta k} + \frac{k\lambda^2}{1+k\lambda^2}. \tag{1,8}$$

are valid.

The general solution of equation (1,4) can be represented in the form

$$\chi(x) = \sum_{i=1}^{6} \overline{C}_i e^{\alpha_i x} = C_1 \sin\lambda x + C_2 \cos\lambda x + C_3 \operatorname{sh}\mu x + C_4 \operatorname{ch}\mu x + C_5 \operatorname{sh}\nu x + C_6 \operatorname{ch}\nu x \tag{1,9}$$

(where $\overline{C}_i, C_i$ are constants).

§ 2. Let us consider a number of boundary-value problems.

A. Hinged edges. The boundary conditions at $x = 0,\ x = 1$ will be

$$\chi = \nabla^2\chi = \nabla^2\nabla^2\chi = 0. \tag{2,1}$$

Substituting (1,9) into (2,1), we arrive at a homogeneous system of linear algebraic equations for the coefficients $C_i(\bar{C}_i)$. The condition for a nonzero solution has the form

$$\Delta = \operatorname{sh} a_1 \operatorname{sh} a_3 \operatorname{sh} a_5, \tag{2,2}$$

where Δ is the determinant composed of the coefficients of $\bar{C}_i$ in the indicated system.

Consequently, one of the roots is $a_1 = in\pi$.

Fig. 1. Variation of the dimensionless vibration frequency ω_* as a function of the shear parameter k for hinged support of the ends without diaphragms

Fig. 1 and Fig. 2

Figure 1: Fig. 1 and Fig. 2

(a) and with absolutely rigid diaphragms at the ends (b); the first five modes; $\vartheta = 0.01$

Fig. 2. Variation of the first mode shape of a three-layer beam as a function of the shear parameter k for hinged ends (a) (diaphragms absent); one end clamped and the other hinged (b)

Substituting a_1 into (1,7), we obtain an explicit expression for the vibration frequency (Fig. 1)

$$\omega_*^2 = n^4\pi^4 \frac{1+\vartheta k n^2\pi^2}{1+kn^2\pi^2}. \tag{2,3}$$

To calculate the vibration modes we shall use the first five relations (2,1). We obtain a system of linear algebraic equations for the coefficients C_i. Solving it, we find

$$C_1 = 1, \qquad C_i = 0 \quad (i = 2, 3, \dots, 6). \tag{2,4}$$

As a result the dimensionless deflection $w(x)$ is equal to (see Fig. 2a)

$$w(x) = (1 + kn^2\pi^2)\sin n\pi x. \tag{2,5}$$

B. Hinged edges. At the edges there are diaphragms, absolutely flexible out of their plane and absolutely rigid in shear in their plane (Fig. 1, 3). The boundary conditions for $x = 0,\ x = 1$ take the form

$$(\nabla^2\chi)_{,x} = \nabla^2(1-\vartheta k\nabla^2)\chi = (1-k\nabla^2)\chi = 0. \tag{2.6}$$

The determinant Δ in this case will be

$$\begin{aligned}\Delta(\vartheta, k, \omega_*) = &\sum_{i,j,k}(\alpha_k^2-\alpha_i^2)(\alpha_j^2-\alpha_i^2)\alpha_j^3\alpha_k^3\alpha\left[1-\vartheta k(\alpha_k^2+\alpha_i^2)+\vartheta k^2\alpha_i^2\alpha_k^2\right]\times\\ &\times\left[1-\vartheta k(\alpha_j^2+\alpha_i^2)+\alpha_i^2\alpha_j^2\right]\operatorname{sh}\alpha_i\,(1-\operatorname{ch}\alpha_j\operatorname{ch}\alpha_k)+\\ &+\frac{1}{2}\operatorname{sh}\alpha_1\operatorname{sh}\alpha_3\operatorname{sh}\alpha_5\sum_{i,j,k}(\alpha_k^2-\alpha_j^2)\alpha_j^6\left[1-\vartheta k(\alpha_k^2+\alpha_j^2)+\vartheta k^2\alpha_j^2\alpha_k^2\right]=0,\end{aligned} \tag{2.7}$$

where the indices i, j, k $(1, 3, 5)$ are permuted cyclically three times in the sums. Joint solution of (1.6) and (2.6) makes it possible to determine the dependence of the vibration frequency ω_* on the parameters $\vartheta,\ k$. The results of the calculations are presented in Fig. 1. The vibration modes are calculated by the method indicated above. In this case all the coefficients C_i are different from zero.

Fig. 3. Dependence of the roots α_i of the frequency equation on the shear parameter k for hinged support of the edges with absolutely rigid diaphragms: first six modes; $\vartheta = 0.01$

Fig. 4. Variation of the dimensionless vibration frequency ω_* as a function of the shear parameter k; both ends of the beam are freely clamped; first eight modes; $\vartheta = 0.01$

C. Free clamping along the edges $x = 0,\ x = 1$ (Fig. 4). The boundary conditions are

$$(1 - k\nabla^2)\chi = \chi_{,xx} = (\nabla^2\chi)_{,xx} = 0. \tag{2.8}$$

The determinant Δ is equal to

$$\begin{aligned}\Delta(\vartheta, k, \omega_*) = &\sum_{i,j,k} \alpha_i^2\alpha_j\alpha_k(\alpha_k^2 - \alpha_i^2)(\alpha_j^2 - \alpha_i^2)\times \\ &\times \left[1 - k(\alpha_k^2 + \alpha_j^2) + k^2\alpha_j^2\alpha_k^2\right] \operatorname{sh}\alpha_i\,(1 - \operatorname{ch}\alpha_j \operatorname{ch}\alpha_k)+ \\ &+ \frac{1}{2}\operatorname{sh}\alpha_1 \operatorname{sh}\alpha_3 \operatorname{sh}\alpha_5 \sum_{i,j,k} \alpha_i^2\alpha_k^2(\alpha_k^2 - \alpha_i^2)^2(1 - k\alpha_j^2)^2\end{aligned} \tag{2.9}$$

$(i, j, k$ —threefold cyclic permutation).

7. Free clamping along the edge $x = 0$ and hinged support along the edge $x = 1$. The boundary conditions are

$$\chi_{,xx} = (\nabla^2\chi)_{,xx} = (1 - k\nabla^2)\chi = 0 \qquad (x = 0); \tag{2.10}$$

$$\chi = \nabla^2\chi = \nabla^2\nabla^2\chi = 0 \qquad (x = 1). \tag{2.11}$$

The conditions for determining the vibration frequencies have the form

$$\Delta(\vartheta, k, \omega_*) = \sum_{i,j,k} \alpha_k\alpha_j(\alpha_k^2 - \alpha_j^2)^2(\alpha_k^2 - \alpha_i^2)(\alpha_j^2 - \alpha_i^2)(1 - k\alpha_i^2)\times$$

$$\times \operatorname{sh}\alpha_i\ \operatorname{ch}\alpha_j\ \operatorname{ch}\alpha_k = 0 \tag{2.12}$$

(where the indices i, j, k $(1, 3, 5)$ are cyclically permuted three times). For the coefficients C_i of the vibration modes, compact expressions in terms of the roots α_i are obtained in this case (Fig. 2b).

D. Free clamping along the edge $x = 0$; the edge $x = 1$ is free. The boundary conditions are

$$(1 - k\nabla^2)\chi = \chi_{,xx} = (\nabla^2\chi)_{,xx} = 0 \qquad (x = 0); \tag{2.13}$$

$$\nabla^2\chi = \nabla^2\nabla^2\chi = (1 - \vartheta k\nabla^2)\nabla^2\chi_{,xx} = 0 \qquad (x = 1). \tag{2.14}$$

The condition for determining the vibration frequencies takes the form

$$\Delta(\vartheta, k, \omega_*) = \sum_{i,j,k} \alpha_i^2(\alpha_i^2 - \alpha_j^2)(\alpha_k^2 - \alpha_i^2)\alpha_i\alpha_j\alpha_k \times$$

$$\times \left[\alpha_j^4(1 - \vartheta k\alpha_j^2)(1 - k\alpha_k^2) + \alpha_k^4(1 - \vartheta k\alpha_k^2)(1 - k\alpha_j^2)\right] \operatorname{ch}\alpha_i +$$

$$+ \sum_{i,j,k} \alpha_i^2(\alpha_i^2 - \alpha_j^2)(\alpha_k^2 - \alpha_i^2)\alpha_i\alpha_j\alpha_k \times \tag{2.15}$$

$$\times \left[\alpha_j^4(1 - \vartheta k\alpha_j^2)(1 - k\alpha_k^2) + \alpha_k^4(1 - \vartheta k\alpha_k^2)(1 - k\alpha_j^2)\right] \operatorname{ch}\alpha_i \operatorname{sh}\alpha_j \operatorname{sh}\alpha_k +$$

$$+\alpha_1^3\alpha_3^3\alpha_5^3 \operatorname{ch}\alpha_1 \operatorname{ch}\alpha_3 \operatorname{ch}\alpha_5 \sum_{i,j,k} (\alpha_i^2 - \alpha_j^2)(1 - \vartheta k\alpha_k^2)(1 - k\alpha_k^2) = 0$$

(the indices i, j, k $(1, 3, 5)$ are cyclically permuted three times).

E. Free clamping along the edge $x = 0$; on the free edge $x = 1$ there is a diaphragm, absolutely flexible out of its plane and absolutely rigid in shear in its plane. The edge conditions will be

$$(1 - k\nabla^2)\chi = \chi_{,xx} = (\nabla^2\chi) = 0 \qquad (x = 0); \tag{2.16}$$

$$\nabla^2\chi_{,xx} = \nabla^2(1 - \vartheta k\nabla^2)\chi = \nabla^2\nabla^2\chi_{,xx} = 0 \qquad (x = 1). \tag{2.17}$$

The determinant is equal to

$$\Delta(\vartheta, k, \omega_*) = \sum_{i,j,k} \alpha_i^3(\alpha_j^3 - \alpha_i^2)(\alpha_k^2 - \alpha_i^2)\alpha_i\alpha_j\alpha_k \times$$

$$\times\left[\alpha_j^4(1-\vartheta k\alpha_j^2)(1-k\alpha_k^2)+\alpha_k^4(1-\vartheta k\alpha_k^2)(1-k\alpha_j^2)\right]\operatorname{sh}\alpha_i+$$

$$+\sum_{i,j,k}\alpha_i(\alpha_i^2-\alpha_j^2)(\alpha_k^2-\alpha_i^2)\alpha_i^3\alpha_j^3\alpha_k^3\times$$

$$\times\left[(1-\vartheta k\alpha_j^2)(1-k\alpha_k^2)+(1-\vartheta k\alpha_k^2)(1-k\alpha_j^2)\right]\operatorname{sh}\alpha_i\ \operatorname{ch}\alpha_j\ \operatorname{ch}\alpha_k+$$

$$+\alpha_1^2\alpha_3^2\alpha_5^2\operatorname{sh}\alpha_1\ \operatorname{sh}\alpha_3\ \operatorname{sh}\alpha_5\sum_{i,j,k}\alpha_i^2\alpha_j^2(\alpha_i^2-\alpha_j^2)^2(1-\vartheta k\alpha_k^2)(1-k\alpha_k^2)=0 \quad (2.18)$$

(the indices i, j, k $(1, 3, 5)$ are cyclically permuted under the summation signs three times).

All computations were carried out on the Strela electronic computer. In this, Ya. I. Alikhatkin rendered the author great assistance.

The author expresses gratitude to E. I. Grigolyuk for a number of important suggestions.

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and Technical Information

Received
16 III 1966

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Note: Figure translations are in progress. See original paper for figures.