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MATHEMATICAL PHYSICS

1967

SovietRxiv

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UDC 534.26

MATHEMATICAL PHYSICS

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ON THE UNIQUENESS OF THE SOLUTION OF INTEGRAL EQUATIONS (OF THE SECOND KIND) FOR DIFFRACTION PROBLEMS ON OPEN SCREENS

(Presented by Academician V. A. Fock on 20 II 1967)

In paper ([1]) a method was given for reducing problems of diffraction of waves on open infinitely thin screens to integral equations of the second kind

$$w(q) = \int_S K(g,q)w(g)\,dg + \Phi(q), \qquad q \in S. \tag{1}$$

Here S is the surface of the screen (in the general case consisting of several sufficiently smooth surfaces); q and g are points on S; dg is the element of area at the point g; w is the unknown "current" on S; K and Φ are prescribed functions. The solution $w(q)$ is sought in the class of functions continuous on the open surface S, whose behavior as one approaches the edge of S corresponds to Meixner' s edge conditions for the corresponding problem.

It follows from the derivation of equation (1) that the current w on S of the original diffraction problem satisfies it. Therefore, taking into account that the latter is uniquely solvable, in order to prove the equivalence of the problem of solving equation (1) and the original problem it is sufficient to show that (1) has only one solution. For this it is necessary to prove that the corresponding homogeneous equation

$$w(q) = \int_S K(g,q)w(g)\,dg, \qquad q \in S, \tag{2}$$

has only the zero solution.

§ 1. **Scalar problem with Dirichlet boundary conditions.** If the primary field $\psi_0(q)$ is diffracted by S, whose edge is the contour L, then the secondary field $\psi(q)$ satisfies the equations*

$$(\nabla^2 + k^2)\psi = 0, \qquad \psi^+ = \psi^- = -\psi_0 \quad \text{on } S, \tag{3}$$

the radiation principle and Meixner' s conditions on L. For this problem (see ([1]), (9) and (10))

$$\Phi(q) = -4\,\frac{\partial}{\partial n_q}\left(\int_S \psi_0(p)\frac{\partial f(q,p)}{\partial n_p}\,dp\right)^{\pm}, \qquad w = \frac{\partial \psi^+}{\partial n} - \frac{\partial \psi^-}{\partial n}; \tag{4}$$

$$K(g,q) = 4\,\frac{\partial}{\partial n_q}\left(\int_S \frac{\partial f(q,p)}{\partial n_p} f(p,g)\,dp\right)^{\pm}, \qquad f(q,p) = \frac{1}{4\pi}\frac{e^{-ik|p-q|}}{|p-q|}. \tag{5}$$

Let us prove that (2), with Φ and K defined by expressions (4) and (5), has only the zero solution.

Arguing by contradiction, we shall assume that (2) has some solution $w_0(q)$. Introduce the function

$$\varphi(q) \equiv -\int_S w_0(p) f(p,q)\,dp. \tag{6}$$

* ψ^+, $\partial\psi^+/\partial n$ and ψ^-, $\partial\psi^-/\partial n$ are the limiting values when passing to S from the side toward which n is directed, and from the opposite side. If they are the same, we shall sometimes write $F^{\pm}$, $\partial F^{\pm}/\partial n$.

It obviously satisfies the equation $(\nabla^2 + k^2)\varphi = 0$ and the condition $\varphi^+ = \varphi^- = \varphi$ on S (cf. (3))*. Therefore, repeating the transformations carried out in (1) in order to obtain (1), we find

$$w_0(q) = \int_S K(g,q) w_0(g)\,dg + \Phi_1(q), \qquad q \in S. \tag{7}$$

Here Φ_1 differs from Φ by the replacement of $\varphi_0(p)$ by $-\varphi(p)$, since this is precisely the difference between the boundary conditions for ψ and φ on S (see above). Since $w_0(q)$, by definition, satisfies equation (2), it follows from (7) that (see (4))

$$\Phi_1(q) \equiv 4\frac{\partial}{\partial n_q}\left(\int_S \varphi(p)\frac{\partial f(q,p)}{\partial n_p}\,dp\right)^{\pm} = 0, \qquad q \in S. \tag{8}$$

Let us now consider the function

$$V(q) \equiv \int_S \varphi(p) \frac{\partial f(q,p)}{\partial n_p}\, dp. \tag{9}$$

For it, as follows from (8), the conditions $\partial V^+/\partial n = \partial V^-/\partial n = 0$ on S hold. Thus V is a wave double-layer potential with zero boundary condition of Neumann type on S, and by the uniqueness theorem $V(q) \equiv 0$. Hence it follows that also $\varphi(p) = 0$ on S, since the jump of (9) upon crossing S is equal to φ at the point of crossing. The function $\varphi(q)$ is a wave single-layer potential (see (6)) and, as has just been shown, satisfies the boundary conditions $\varphi^+ = \varphi^- = \varphi = 0$ on S. Therefore, by the uniqueness theorem, $\varphi(q) \equiv 0$ everywhere. From the latter it immediately follows that $w_0(q) = 0$ on S, for $w_0(q)$ is equal to the jump of the normal derivative of (6) upon crossing S. Thus, (2) has no solution distinct from zero.

§ 2. **Scalar problem with Neumann boundary conditions.** This problem differs from the preceding one only by replacing the conditions (3) by the following:

$$\partial\psi^+/\partial n = \partial\psi^-/\partial n = -\partial\psi_0/\partial n \quad \text{on } S \tag{10}$$

and by the fact that now $w = \psi^+ - \psi^-$. Equation (1) for this problem has a kernel differing from the kernel (5) only by interchange of the arguments, and a free term of the form (see (1′))

$$\Phi(q) = 4 \int_S \frac{\partial \psi_0}{\partial n_p} f(q,p)\, dp. \tag{11}$$

Assuming that (2) for the problem under consideration has a solution $w_0(q)$, introduce the function

$$\varphi(q) \equiv \int_S w_0(p) \frac{\partial f(q,p)}{\partial n_p}\, dp. \tag{12}$$

$\varphi(q)$ is a wave double-layer potential and, consequently, satisfies the Helmholtz equation and the condition

$$\partial\varphi^+/\partial n = \partial\varphi^-/\partial n = \partial\varphi/\partial n \quad \text{on } S. \tag{13}$$

Therefore, for w_0 equation (1) is also valid, with the difference that in the free term (11) $\partial\psi_0/\partial n_p$ is replaced by $-\partial\varphi/\partial n_p$, in accordance with the analogous difference between (10) and (13). Since, moreover, w_0, by definition, satisfies (2), the corresponding free term (see (11)) must vanish,

$$-4\int_S \frac{\partial\varphi}{\partial n_p} f(q,p)\,dp = 0 \quad \text{on } S. \tag{14}$$

* The function $\varphi(q)$ also satisfies the radiation principle and the Meixner condition on L. This will always be kept in mind below when considering wave single- and double-layer potentials.

We now introduce the function

$$V(q) = \int_S \frac{\partial\varphi}{\partial n_p} f(q,p)\,dp, \tag{15}$$

which is a wave potential of a simple layer and which, taking (14) into account, satisfies the condition $V^+ = V^- = 0$ on S.

On the basis of the corresponding uniqueness theorem it follows from this that $V(q) \equiv 0$ everywhere. From the latter we conclude that $\partial\varphi/\partial n = 0$ on S, since the jump of the normal derivative of (15) on passing through S is equal to $\partial\varphi/\partial n$. The function $\varphi(q)$ (see (12)) is a wave potential of a double layer and, as has just been shown, satisfies (see (13)) the condition

$$\partial\varphi^+/\partial n = \partial\varphi^-/\partial n = \partial\varphi/\partial n = 0 \quad \text{on } S.$$

Therefore, on the basis of the uniqueness theorem, $\varphi(q) \equiv 0$ everywhere. It follows from this that $w_0(q) = 0$ on S, since w_0 is equal to the jump of $\varphi(q)$ (see (12)) on passing through S. Consequently, for the present problem as well, (2) has no solution different from zero.

§ 3. **Scalar problem with Dirichlet boundary conditions; an integral equation with a kernel connected with Green's functions.** This problem, identical to that considered in § 1, can also be reduced to equation (1) with kernel and free term (see (1) (18a) and (21a))

$$K(g,q) = -\int_\Sigma \frac{\partial^2(G^i+G^e)}{\partial n_p \partial n_q} f(g,p)\,dp, \qquad \Phi(q) = -\frac{\partial}{\partial n_q}\left(\int_S \psi_0(p)\,\frac{\partial(G^i+G^e)}{\partial n_p}\,dp\right)^{\pm}. \tag{16}$$

Here Σ is some surface completing S to a closed one; in particular, it may go off to infinity. The surface $S+\Sigma$ divides space into two domains v_i (toward which $\mathbf{n}$ is directed) and v_e. $G^i(p,q)$ and $G^e(p,q)$ are Green' s functions of the Helmholtz equation for these domains with zero boundary conditions. Assuming that in this case too (2) has a solution $w_0(q)$, we construct a function of the type (6). Then, repeating the arguments of § 1, we arrive at an equality of the type (8), which now, taking into account expression (16) for the free term, is written as follows:

$$\frac{\partial}{\partial n_q}\left(\int_S \varphi(p)\,\frac{\partial(G^i+G^e)}{\partial n_p}\,dp\right)^{\pm}=0, \qquad q\in S. \tag{17}$$

Define in v_i+v_e the function $V(q)$ by means of the equalities

$$V(q)=\int_S \varphi(p)\frac{\partial G^i}{\partial n_p}\,dp, \qquad q\in v_i; \qquad V(q)=-\int_S \varphi(p)\frac{\partial G^e}{\partial n_p}\,dp, \qquad q\in v_e. \tag{18}$$

It obviously satisfies in v_i+v_e the Helmholtz equation and the conditions

$$V^+=V^-=0 \text{ on } \Sigma, \qquad V^+=V^- \text{ on } S, \qquad \partial V^+/\partial n=\partial V^-/\partial n \text{ on } S.$$

The first two follow from the fact that G^i and G^e are Green' s functions for the domains v_i and v_e, and the last follows from (17). Therefore, taking the uniqueness theorem into account, $V(q)\equiv 0$ everywhere. On the other hand, from (18) it follows that $V^+=V^-=\varphi$ on S; hence $\varphi(q)=0$ for $q\in S$. Repeating now the arguments of the last section of § 1, we make certain that also $w_0(q)=0$ on S, i.e., in the case under consideration as well, (2) has only the zero solution.

§ 4. **Diffraction of an electromagnetic wave.** For this problem the current $\mathbf{w}$ on S satisfies a vector equation of the type (1), in which

the kernel is a tensor (see ([1]), (33)), and the free term has the form*

$$\Phi(q)=\frac{1}{i\omega}\left[\mathbf{n}_q\int_S\{\mathbf{H}^i(q,p;\mathbf{P})+\mathbf{H}^e(q,p;\mathbf{P})\}\,dp\right]. \tag{19}$$

Here $\mathbf{P}=[\mathbf{n}_p\mathbf{E}^0(p)]$; $\mathbf{E}^0$ is the electric vector of the primary wave incident on S; $\mathbf{H}^i(q,p;\mathbf{P})$ and $\mathbf{H}^e(q,p;\mathbf{P})$ are the magnetic vectors of the auxiliary fields excited in v_i and v_e (see § 3), respectively; their three arguments denote: q, the observation point; p, the point where the source is located–a magnetic dipole with moment $\mathbf{P}$. In computing these fields the surface $S+\Sigma$ is assumed to be perfectly conducting. Assuming again that (2) has the solution w_0, introduce the vector

$$\vec{\mathcal{E}}(q)=\frac{1}{i\omega\varepsilon}\int_S\{k^2w_0(g)-(\mathbf{w}_0(g)\nabla_g)\nabla_p\}f(g,p)\,dg. \tag{20}$$

For it, obviously, the conditions $[\mathbf{n}\vec{\mathcal{E}}^+]=[\mathbf{n}\vec{\mathcal{E}}^-]=[\mathbf{n}\vec{\mathcal{E}}]$ on S are valid. Therefore, repeating the arguments of § 4 of paper ([1]), one can show that w_0 satisfies (1), with the difference that in the free term (19) $\mathbf{E}^0$ must be replaced by $-\vec{\mathcal{E}}$. Taking into account, moreover, that w_0 satisfies (2), we obtain (see (19)) the equality

$$\left[\mathbf{n}_q \int_S \{\mathbf{H}^i(q,p;\mathbf{P}_1) + \mathbf{H}^e(q,p;\mathbf{P}_1)\}\,dp\right] = 0, \quad q \in S, \quad \mathbf{P}_1 = -[\mathbf{n}_p \vec{\mathcal{E}}(p)]. \quad (21)$$

Let us now define in $v_i + v_e$ the field $\mathbf{e}, \mathbf{h}$:

$$\mathbf{e}(q) = \int_S \mathbf{E}^i(q,p;\mathbf{P}_1)\,dp, \quad \mathbf{h}(q) = \int_S \mathbf{H}^i(q,p;\mathbf{P}_1)\,dp, \quad q \in v_i; \quad (22)$$

$$\mathbf{e}(q) = -\int_S \mathbf{E}^e(q,p;\mathbf{P}_1)\,dp, \quad \mathbf{h}(q) = -\int_S \mathbf{H}^e(q,p;\mathbf{P}_1)\,dp, \quad q \in v_e$$

It obviously satisfies the boundary conditions

$$[\mathbf{ne}^+] = [\mathbf{ne}^-] = 0 \text{ on } \Sigma, \qquad [\mathbf{ne}^+] = [\mathbf{ne}^-] \text{ on } S, \qquad [\mathbf{nh}^+] = [\mathbf{nh}^-] \text{ on } S.$$

The first two are a consequence of the special choice of the fields $\mathbf{E}^i, \mathbf{H}^i$ and $\mathbf{E}^e, \mathbf{H}^e$, and the last follows from (21). Therefore, on the basis of the uniqueness theorem, the field $\mathbf{e}(q) = \mathbf{h}(q) \equiv 0$ everywhere. Hence we immediately conclude that $[\mathbf{n}\vec{\mathcal{E}}] = 0$ on S, since from the upper line of (22) it follows ([2]) that $[\mathbf{ne}^+] = -[\mathbf{ne}^-] = -i\omega\mathbf{P}_1 = i\omega[\mathbf{n}\vec{\mathcal{E}}]$ on S. $\vec{\mathcal{E}}$ (see (20)) is the electric vector of a field regular everywhere outside S and satisfying the condition $[\mathbf{n}\vec{\mathcal{E}}] = 0$ on S. Consequently, this field is identically equal to zero, and hence the current $w_0(q) = 0$ on S as well, since it coincides with the jump of the magnetic vector in passing through S.

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Received
13 II 1967

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* Formulas (31) from ([1]) are easily reduced to the form (19).

Note: Figure translations are in progress. See original paper for figures.