Complex eigenvalues of a non-selfadjoint operator

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Abstract

Full Text

Preamble

ON THE COMPLEX EIGENVALUES OF A NON-SELF-ADJOINT OPERATOR

M. GEKHTMAN, A. G. KOSTYUCHENKO

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Consider the self-adjoint differential expression

$$l(y) = (-1)^m y^{(2m)} + (p_1(x) y^{(m-1)})^{(m-1)} + \cdots + p_m(x) y, \qquad (1)$$

whose coefficients are real-valued functions possessing continuous derivatives up to the required order. Furthermore, we shall assume that these coefficients are such that for any compactly supported, sufficiently smooth functions, the following inequality holds:

$$\int_{-\infty}^{\infty} l(f)\overline{f(x)}dx \geq 0. \qquad (2)$$

As is well known, the differential expression $l(y)$ generates a symmetric operator in $L_2(-\infty, \infty)$, which we shall denote by L_0. This operator admits self-adjoint extensions. Let us fix one such self-adjoint extension of the operator L_0 and denote it hereafter by L. By virtue of condition (2), the spectrum of the operator L is bounded from below; without loss of generality, we may assume that the spectrum of L is non-negative.

Now, let $q(x)$ be a continuous real-valued function defined on the entire real axis and satisfying the condition $q(x) \to 0$ as $|x| \to \infty$. We further consider the non-self-adjoint differential operator T in $L_2(-\infty, \infty)$ defined by $Tu =$

$Lu + iq(x)u$. It is known that the limit points of the spectrum of the operator under consideration can only lie on the real axis within the spectral domain of the unperturbed self-adjoint operator L. This fact was first established by M. A. Naimark [?] for the Sturm-Liouville operator, and subsequently for the Schrödinger operator by I. M. Gelfand [?]. We note that Gelfand' s proof is of a general character: in proving that the limit points of the spectrum of the perturbed operator must lie on the real axis, he utilizes only the fact that the kernel of the resolvent of the unperturbed operator is of Carleman type, i.e., $|K(x, y; z)| < C$ when z varies within a bounded region. In the case of self-adjoint differential operators generated by ordinary differential equations, the resolvent kernel is always of Carleman type (see, for example, [?]).

It is known that the entire spectrum of the operator T is located within the strip $|\mathrm{Im}\ z| \leq \max |q(x)|$ of the complex z-plane ($z = \sigma + i\tau$). The purpose of this note is to refine this fact. Specifically, we show that as $\sigma \to \infty$, the spectrum of the operator T lies in the region $|\tau| \leq C\sigma^{-\alpha}$, where C is a constant and $\alpha > 0$. In the case where $q(x)$ is a non-negative function satisfying certain conditions, a similar assertion for the non-self-adjoint Sturm-Liouville operator was previously proven in [?]. Since the limit spectrum of the operator T is located only on the real axis, as noted above, it is sufficient to show that all eigenvalues of the operator T are situated in the region $|\tau| < K(\sigma)$.

Let us first consider the case where $q(x)$ is a compactly supported function vanishing outside a finite interval. Suppose $z = \sigma + i\tau$ is a complex eigenvalue of the operator T, and u is the corresponding eigenfunction, i.e.,

Lu 0 –z 0 u = –iq(x)u 0 . (5)

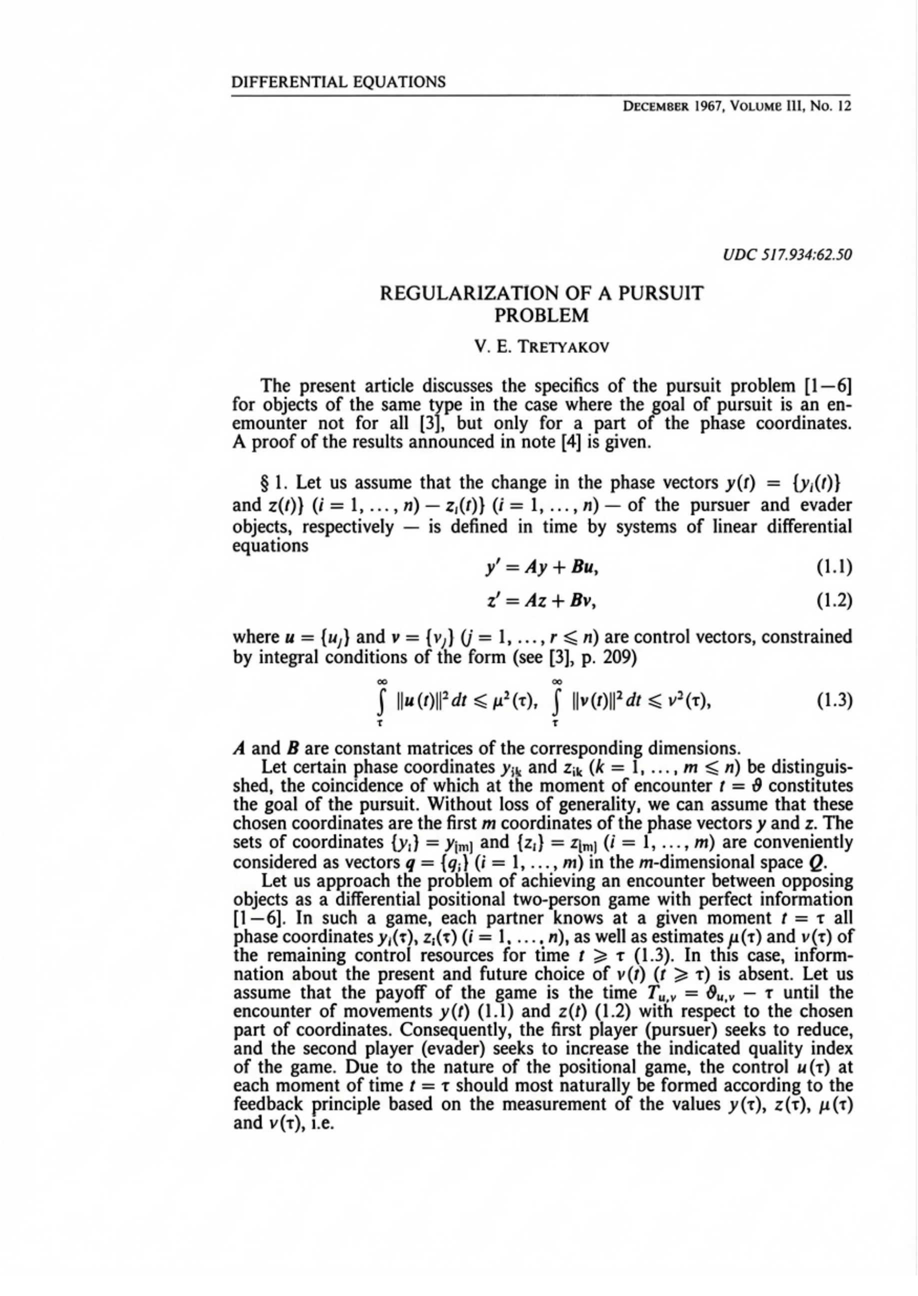
Let us denote the resolvent of the operator by R_λ. Since the condition $q(x)u \in L_2$ holds, we can apply the resolvent to both sides of the equality. Consequently, instead of the differential equation, we obtain an equivalent integral equation.

and $u_0 = -R_\lambda q(x)u_0$. (6)

It is well known that the operator R_λ is an integral operator with a kernel $K(x, y, \lambda)$, where the kernel is expressed through the spectral function $\theta(x, y; \lambda)$ of the operator by the relation $K(x, y, \lambda) = \int_{-\infty}^{\infty} \frac{d\theta(x,y;\lambda)}{\mu-\lambda}$. It was proven by one of the authors of this note [?] that the spectral function $\theta(x, y; \lambda)$ can be represented as $\theta(x, y; \lambda) = \theta_0(x, y; \lambda) + ...$

$$\theta_0(x, y; \lambda) = \begin{cases} \frac{1}{\pi}\frac{\sin\sqrt{\lambda}(x-y)}{x-y}, & \text{if } \lambda > 0, \\ 0, & \text{if } \lambda < 0, \end{cases}$$

Furthermore, the function $\delta(x, y; \lambda)$ tends to zero uniformly in each finite domain of the variables. If we now denote $v(x) = q(x)u_0(x)$ and take into account the finiteness of the function $q(x)$ on the interval $[a, b]$, we obtain an integral equation for determining the complex eigenvalues and eigenfunctions of the operator.



DIFFERENTIAL EQUATIONS

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REGULARIZATION OF A PURSUIT PROBLEM

V. E. TRETYAKOV

The present article discusses the specifics of the pursuit problem [1—6] for objects of the same type in the case where the goal of pursuit is an enemounter not for all [3], but only for a part of the phase coordinates. A proof of the results announced in note [4] is given.

§ 1. Let us assume that the change in the phase vectors $y(t) = \{y_i(t)\}$ and $z(t)\}$ $(i = 1, \dots, n) - z_i(t)\}$ $(i = 1, \dots, n)$ — of the pursuer and evader objects, respectively — is defined in time by systems of linear differential equations

$$y' = \boldsymbol{A}y + \boldsymbol{B}\boldsymbol{u}, \tag{1.1}$$

$$z' = \boldsymbol{A}z + \boldsymbol{B}\boldsymbol{v}, \tag{1.2}$$

where $\boldsymbol{u} = \{u_j\}$ and $\boldsymbol{v} = \{v_j\}$ $(j = 1, \dots, r \leqslant n)$ are control vectors, constrained by integral conditions of the form (see [3], p. 209)

$$\int_{\tau}^{\infty} \|\boldsymbol{u}(t)\|^2\, dt \leqslant \mu^2(\tau), \quad \int_{\tau}^{\infty} \|\boldsymbol{v}(t)\|^2\, dt \leqslant \nu^2(\tau), \tag{1.3}$$

$\boldsymbol{A}$ and $\boldsymbol{B}$ are constant matrices of the corresponding dimensions.

Let certain phase coordinates y_{jk} and z_{ik} $(k = 1, \dots, m \leqslant n)$ be distinguisshed, the coincidence of which at the moment of encounter $t = \vartheta$ constitutes the goal of the pursuit. Without loss of generality, we can assume that these chosen coordinates are the first m coordinates of the phase vectors y and z. The sets of coordinates $\{y_i\} = y_{[m]}$ and $\{z_i\} = z_{[m]}$ $(i = 1, \dots, m)$ are conveniently considered as vectors $\boldsymbol{q} = \{q_i\}$ $(i = 1, \dots, m)$ in the m-dimensional space $\boldsymbol{Q}$.

Let us approach the problem of achieving an encounter between opposing objects as a differential positional two-person game with perfect information [1—6]. In such a game, each partner knows at a given moment $t = \tau$ all phase coordinates $y_i(\tau)$, $z_i(\tau)$ $(i = 1, \dots, n)$, as well as estimates $\mu(\tau)$ and $\nu(\tau)$ of the remaining control resources for time $t \geqslant \tau$ (1.3). In this case, informnation about the present and future choice of $v(t)$ $(t \geqslant \tau)$ is absent. Let us assume that the payoff of the game is the time $T_{u,v} = \vartheta_{u,v} - \tau$ until the encounter of movements $y(t)$ (1.1) and $z(t)$ (1.2) with respect to the chosen part of coordinates. Consequently, the first player (pursuer) seeks to reduce, and the second player (evader) seeks to increase the indicated quality index of the game. Due to the nature of the positional game, the control $u(\tau)$ at each moment of time $t = \tau$ should most naturally be formed according to the feedback principle based on the measurement of the values $y(\tau)$, $z(\tau)$, $\mu(\tau)$ and $v(\tau)$, i.e.

Figure 1: Figure 1

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In operator form, let us denote for brevity:

On Complex Eigenvalues of a Single Operator

We shall prove that for sufficiently large values of a (with m fixed), the norm of the operator can be made arbitrarily small. From this, it follows that for such $z = a + i\zeta$, the equation in $L_2(-\infty, \infty)$ can only have a trivial solution. Consequently, the specified values are not eigenvalues of the operator.

The norm of the operator can be easily estimated using the following inequality:

$$\|K\| \leq C\sqrt{\int\int |K(x,y;z)|^2 \, dxdy}$$

(where $C = \max|q(x)|$).

Furthermore, we will demonstrate that the kernel of the resolvent $R(x, y; z)$ tends to zero uniformly with respect to x and y varying within a bounded domain, provided that $a \to \infty$ (with m fixed). It follows from formula (7) that $Q(x, y; \lambda)\cdots$

The kernel function is defined as:

$$K(x,y;z)$$

We transform the integral on the left side of this equality by performing integration by parts in the second term:

$$\int \Gamma\phi(x,y,\lambda)\, dx - z$$

In performing the transformation (12), we utilized the fact that if x and y vary within a bounded domain and $\lambda \to \infty$, then $\phi(x, y, \lambda)$ behaves uniformly with respect to these variables.

We shall now verify that each term in the formula tends to zero as $z \to \infty$ (with τ fixed), uniformly with respect to the variables x and y varying within a bounded domain. Let us denote the integral by $\Phi(x, y, z)$. Since $\Phi(x, y, z)$ is the kernel of the resolvent of the operator $L = (-\Delta)$, it is a decaying solution of the equation:

[The text ends here, but typically follows with the corresponding differential equation for the resolvent kernel.]

$b_0 K^{-1} b = \delta(x - y)$

Using the Fourier transform, we obtain $|x - y|$. By calculating the integral using residues, we find $\Phi(x, y; z)$.

M. M. Gekhtman and A. G. Kostyuchenko. The roots of the equation s (Im $s > 0$). From formula $(*)$, it follows that

$$|\Phi_0(x, y; z)| < \frac{C}{|x - y|}$$

From this estimate, it is clear that $\Phi(x, y; z)$ tends to zero as $z \to \infty$ uniformly with respect to the variables. Let us verify that the last term in formula (12) also tends to zero as $z \to \infty$. Given $\epsilon > 0$, we choose $N(\epsilon) > 0$ such that $|\phi(x, y, \lambda)| < \epsilon$ for $\lambda > N$, uniformly for variables changing within a bounded domain. We transform the integral:

$$\int_0^\infty \dots d\lambda = \dots$$

(13) We now estimate each term in formula (13).

$|J_1| < C$ as $z \to \infty$,

since it is possible to pass to the limit under the integral sign.

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Due to the arbitrariness of $\delta > 0$, we conclude that $\lim \delta = 0$ as $z \to \infty$. Thus, we have shown that each term in formula (12) tends to zero as $z \to \infty$ (for a fixed τ) uniformly with respect to the variables x and y varying within a bounded domain. This implies that $\lim K(x, y) = 0$ as $z \to \infty$ (for fixed τ) uniformly with respect to the variables in a bounded domain. Therefore, in inequality (11), we can pass to the limit under the integral sign, yielding:

$$\lim \|S_z\| = 0 \quad (z \to \infty).$$

It follows from this that on the line $\tau =$ const, starting from a certain point, all points are of regular type for the operator L, and consequently, there are no eigenvalues of the operator on this line beyond that point. We can now prove the following theorem.

Theorem. Let an operator L be defined by the given conditions. Then there exists a non-negative function $K(\sigma)$ such that $\lim K(\sigma) = 0$ as $\sigma \to \infty$, and the entire spectrum of the operator is located within the region defined by $K(\sigma)$.

Proof. If the potential is compactly supported (finite), the theorem is already proven. We shall therefore assume that $q(x)$ satisfies only the general condition. Let $V_0(x) = p(x)$. Since $q(x) \to \infty$, we can represent $p(x)$ as a sum $p(x) = p_1(x) + p_2(x)$, where $p_1(x)$ is compactly supported on $[a, b]$ and $\|p_2\| < \epsilon$. To prove the theorem, we follow the previous reasoning verbatim, which leads to the necessity of showing that:

On the Complex Eigenvalues of a Certain Operator

In the general case, $\lim \|S_z p\| = 0$ as $z \to \infty$ (for fixed τ). Since p_1 is compactly supported, $\lim \|S_z p_1\| = 0$ as $z \to \infty$. As for the remaining terms, the norm of each can be made arbitrarily small due to the presence of the operator p_2 in each term, whose norm satisfies $\|p_2\| < \epsilon$.

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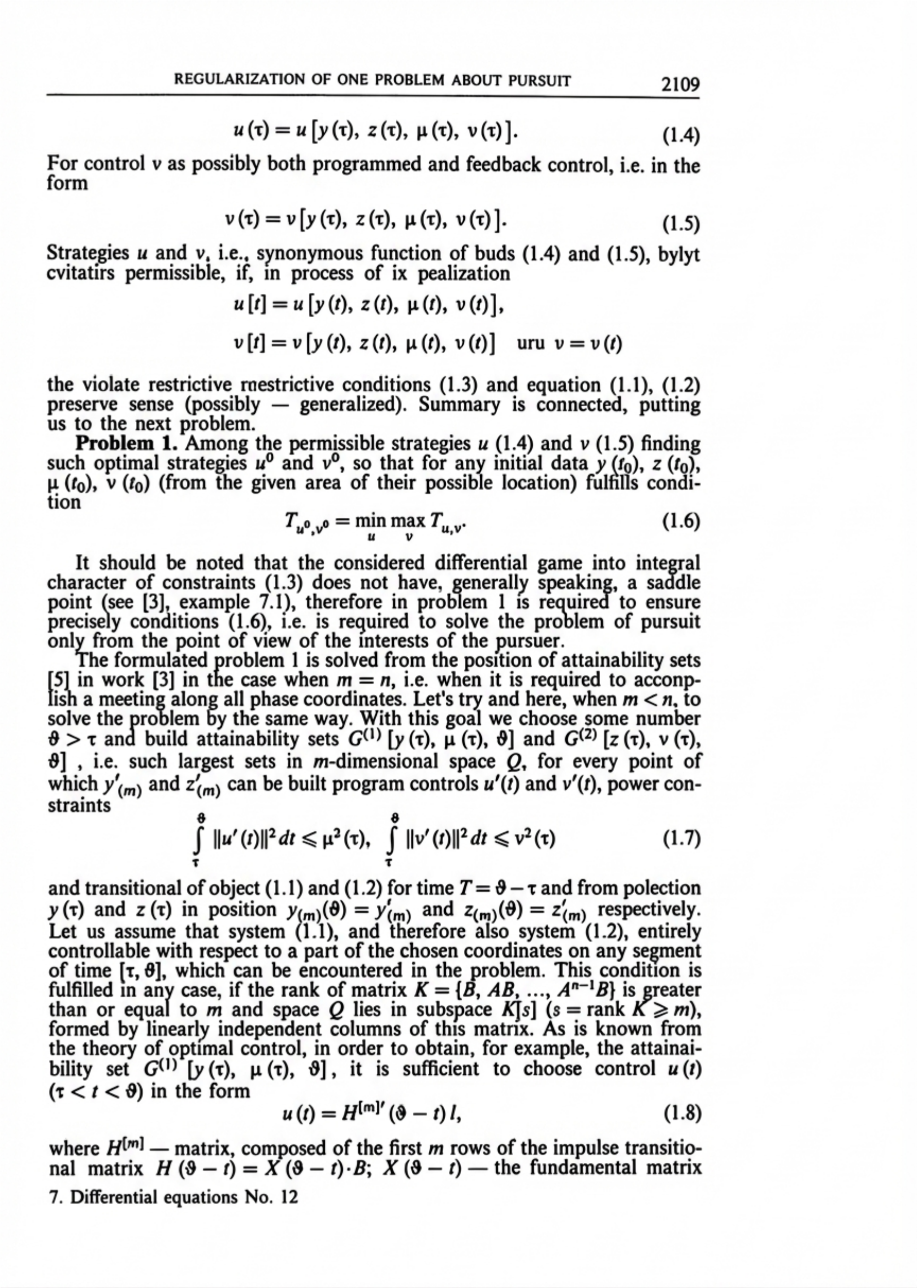
Figures



$$u(\tau) = u\left[y(\tau),\ z(\tau),\ \mu(\tau),\ \nu(\tau)\right]. \tag{1.4}$$

For control v as possibly both programmed and feedback control, i.e. in the form

$$v(\tau) = v\left[y(\tau),\ z(\tau),\ \mu(\tau),\ \nu(\tau)\right]. \tag{1.5}$$

Strategies u and v, i.e., synonymous function of buds (1.4) and (1.5), bylyt cvitatirs permissible, if, in process of ix pealization

$$u[t] = u\left[y(t),\ z(t),\ \mu(t),\ \nu(t)\right],$$

$$v[t] = v\left[y(t),\ z(t),\ \mu(t),\ \nu(t)\right] \quad \text{uru} \ \ v = v(t)$$

the violate restrictive rnestrictive conditions (1.3) and equation (1.1), (1.2) preserve sense (possibly — generalized). Summary is connected, putting us to the next problem.

Problem 1. Among the permissible strategies u (1.4) and v (1.5) finding such optimal strategies u^0 and v^0, so that for any initial data $y(t_0)$, $z(t_0)$, $\mu(t_0)$, $\nu(t_0)$ (from the given area of their possible location) fulfills condition

$$T_{u^0,v^0} = \min_u \max_v T_{u,v}. \tag{1.6}$$

It should be noted that the considered differential game into integral character of constraints (1.3) does not have, generally speaking, a saddle point (see [3], example 7.1), therefore in problem 1 is required to ensure precisely conditions (1.6), i.e. is required to solve the problem of pursuit only from the point of view of the interests of the pursuer.

The formulated problem 1 is solved from the position of attainability sets [5] in work [3] in the case when $m = n$, i.e. when it is required to acconplish a meeting along all phase coordinates. Let's try and here, when $m < n$, to solve the problem by the same way. With this goal we choose some number $\vartheta > \tau$ and build attainability sets $G^{(1)}[y(\tau), \mu(\tau), \vartheta]$ and $G^{(2)}[z(\tau), \nu(\tau), \vartheta]$, i.e. such largest sets in m-dimensional space Q, for every point of which $y'_{(m)}$ and $z'_{(m)}$ can be built program controls $u'(t)$ and $v'(t)$, power constraints

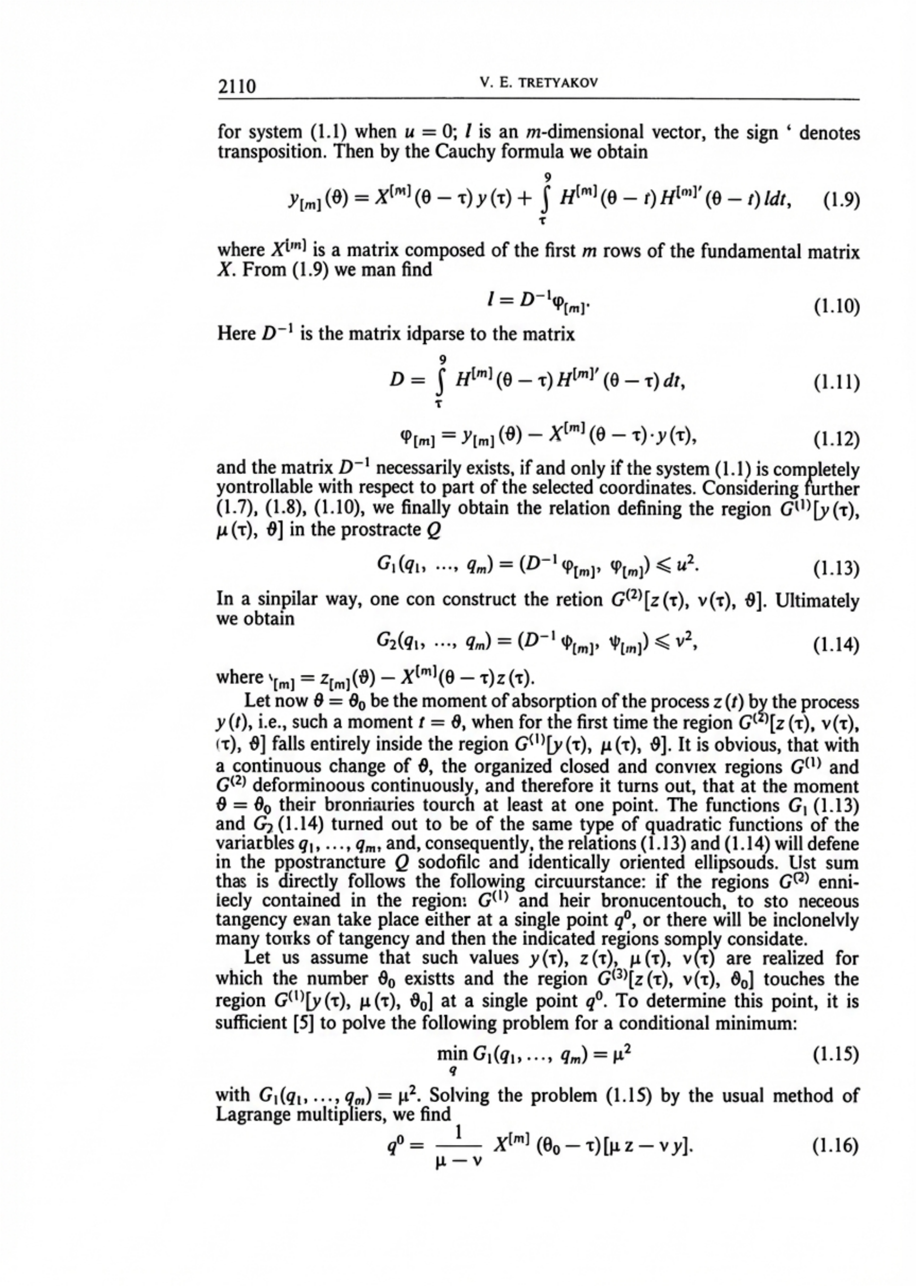
$$\int_\tau^\vartheta \|u'(t)\|^2 dt \leqslant \mu^2(\tau), \quad \int_\tau^\vartheta \|v'(t)\|^2 dt \leqslant \nu^2(\tau) \tag{1.7}$$

and transitional of object (1.1) and (1.2) for time $T = \vartheta - \tau$ and from polection $y(\tau)$ and $z(\tau)$ in position $y_{(m)}(\vartheta) = y'_{(m)}$ and $z_{(m)}(\vartheta) = z'_{(m)}$ respectively. Let us assume that system (1.1), and therefore also system (1.2), entirely controllable with respect to a part of the chosen coordinates on any segment of time $[\tau, \vartheta]$, which can be encountered in the problem. This condition is fulfilled in any case, if the rank of matrix $K = \{B, AB, \ldots, A^{n-1}B\}$ is greater than or equal to m and space Q lies in subspace $K]s]$ ($s = \operatorname{rank} K \geqslant m$), formed by linearly independent columns of this matrix. As is known from the theory of optimal control, in order to obtain, for example, the attainability set $G^{(1)}[y(\tau), \mu(\tau), \vartheta]$, it is sufficient to choose control $u(t)$ ($\tau < t < \vartheta$) in the form

$$u(t) = H^{[m]'}(\vartheta - t)\, l, \tag{1.8}$$

where $H^{[m]}$ — matrix, composed of the first m rows of the impulse transitional matrix $H(\vartheta - t) = X(\vartheta - t)\cdot B$; $X(\vartheta - t)$ — the fundamental matrix

Figure 2: Figure 2



for system (1.1) when $u = 0$; l is an m-dimensional vector, the sign ʻ denotes transposition. Then by the Cauchy formula we obtain

$$y_{[m]}(\theta) = X^{[m]}(\theta - \tau)\, y(\tau) + \int\limits_{\tau}^{9} H^{[m]}(\theta - t)\, H^{[m]\prime}(\theta - t)\, l\,dt, \qquad (1.9)$$

where $X^{[m]}$ is a matrix composed of the first m rows of the fundamental matrix X. From (1.9) we man find

$$l = D^{-1}\varphi_{[m]}. \qquad (1.10)$$

Here D^{-1} is the matrix idparse to the matrix

$$D = \int\limits_{\tau}^{9} H^{[m]}(\theta - \tau)\, H^{[m]\prime}(\theta - \tau)\, dt, \qquad (1.11)$$

$$\varphi_{[m]} = y_{[m]}(\theta) - X^{[m]}(\theta - \tau)\cdot y(\tau), \qquad (1.12)$$

and the matrix D^{-1} necessarily exists, if and only if the system (1.1) is completely yontrollable with respect to part of the selected coordinates. Considering further (1.7), (1.8), (1.10), we finally obtain the relation defining the region $G^{(1)}[y(\tau), \mu(\tau), \vartheta]$ in the prostracte Q

$$G_1(q_1, \ldots, q_m) = (D^{-1}\varphi_{[m]}, \varphi_{[m]}) \leqslant u^2. \qquad (1.13)$$

In a sinpilar way, one con construct the retion $G^{(2)}[z(\tau), \nu(\tau), \vartheta]$. Ultimately we obtain

$$G_2(q_1, \ldots, q_m) = (D^{-1}\psi_{[m]}, \psi_{[m]}) \leqslant \nu^2, \qquad (1.14)$$

where $`_{[m]} = z_{[m]}(\vartheta) - X^{[m]}(\theta - \tau)\, z(\tau)$.

Let now $\theta = \theta_0$ be the moment of absorption of the process $z(t)$ by the process $y(t)$, i.e., such a moment $t = \vartheta$, when for the first time the region $G^{(2)}[z(\tau), \nu(\tau), (\tau), \vartheta]$ falls entirely inside the region $G^{(1)}[y(\tau), \mu(\tau), \vartheta]$. It is obvious, that with a continuous change of θ, the organized closed and conviex regions $G^{(1)}$ and $G^{(2)}$ deforminoous continuously, and therefore it turns out, that at the moment $\vartheta = \theta_0$ their bronriauries tourch at least at one point. The functions G_1 (1.13) and G_2 (1.14) turned out to be of the same type of quadratic functions of the variarbles $q_1, \ldots, q_m$, and, consequently, the relations (1.13) and (1.14) will defene in the ppostrancture Q sodofilc and identically oriented ellipsouds. Ust sum thas is directly follows the following circuurstance: if the regions $G^{(2)}$ enniiecly contained in the region: $G^{(1)}$ and heir bronucentouch, to sto neceous tangency evan take place either at a single point q^0, or there will be inclonelvly many torrks of tangency and then the indicated regions somply considate.

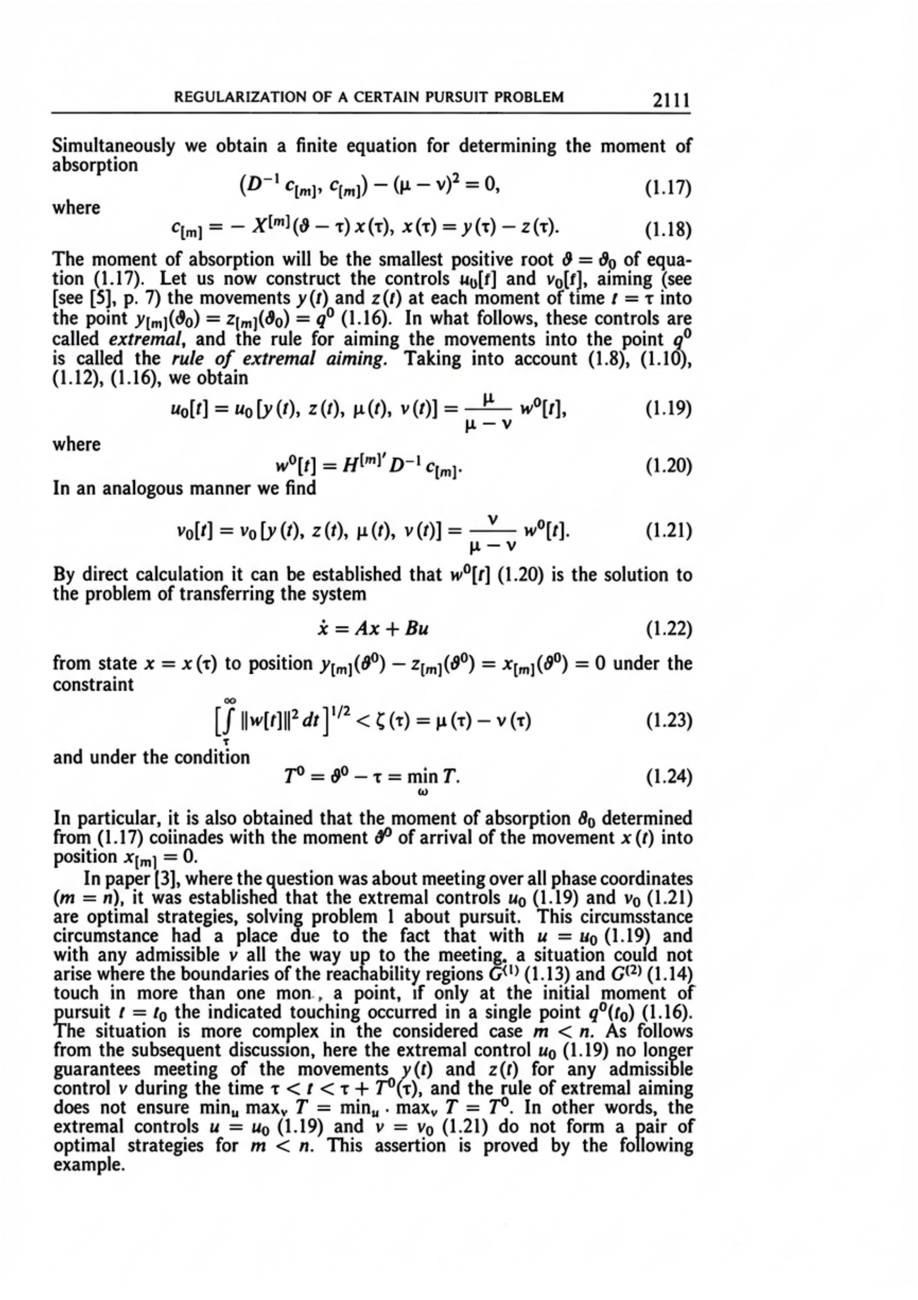
Let us assume that such values $y(\tau)$, $z(\tau)$, $\mu(\tau)$, $\nu(\tau)$ are realized for which the number ϑ_0 existts and the region $G^{(3)}[z(\tau), \nu(\tau), \vartheta_0]$ touches the region $G^{(1)}[y(\tau), \mu(\tau), \vartheta_0]$ at a single point q^0. To determine this point, it is sufficient [5] to polve the following problem for a conditional minimum:

$$\min_q G_1(q_1, \ldots, q_m) = \mu^2 \qquad (1.15)$$

with $G_1(q_1, \ldots, q_m) = \mu^2$. Solving the problem (1.15) by the usual method of Lagrange multipliers, we find

$$q^0 = \frac{1}{\mu - \nu} X^{[m]}(\theta_0 - \tau)[\mu z - \nu y]. \qquad (1.16)$$

Figure 3: Figure 3



Simultaneously we obtain a finite equation for determining the moment of absorption

$$(D^{-1}c_{[m]},\ c_{[m]}) - (\mu - \nu)^2 = 0, \tag{1.17}$$

where

$$c_{[m]} = -\ X^{[m]}(\vartheta - \tau)\, x(\tau),\ \ x(\tau) = y(\tau) - z(\tau). \tag{1.18}$$

The moment of absorption will be the smallest positive root $\vartheta = \vartheta_0$ of equation (1.17). Let us now construct the controls $u_0[t]$ and $v_0[t]$, aiming (see [see [5], p. 7) the movements $y(t)$ and $z(t)$ at each moment of time $t = \tau$ into the point $y_{[m]}(\vartheta_0) = z_{[m]}(\vartheta_0) = q^0$ (1.16). In what follows, these controls are called *extremal*, and the rule for aiming the movements into the point q^0 is called the *rule of extremal aiming*. Taking into account (1.8), (1.10), (1.12), (1.16), we obtain

$$u_0[t] = u_0[y(t),\ z(t),\ \mu(t),\ \nu(t)] = \frac{\mu}{\mu - \nu}\, w^0[t], \tag{1.19}$$

where

$$w^0[t] = H^{[m]\prime} D^{-1} c_{[m]}. \tag{1.20}$$

In an analogous manner we find

$$v_0[t] = v_0[y(t),\ z(t),\ \mu(t),\ \nu(t)] = \frac{\nu}{\mu - \nu}\, w^0[t]. \tag{1.21}$$

By direct calculation it can be established that $w^0[t]$ (1.20) is the solution to the problem of transferring the system

$$\dot{x} = Ax + Bu \tag{1.22}$$

from state $x = x(\tau)$ to position $y_{[m]}(\vartheta^0) - z_{[m]}(\vartheta^0) = x_{[m]}(\vartheta^0) = 0$ under the constraint

$$\Big[\int_{\tau}^{\infty} \|w[t]\|^2\, dt\Big]^{1/2} < \zeta(\tau) = \mu(\tau) - \nu(\tau) \tag{1.23}$$

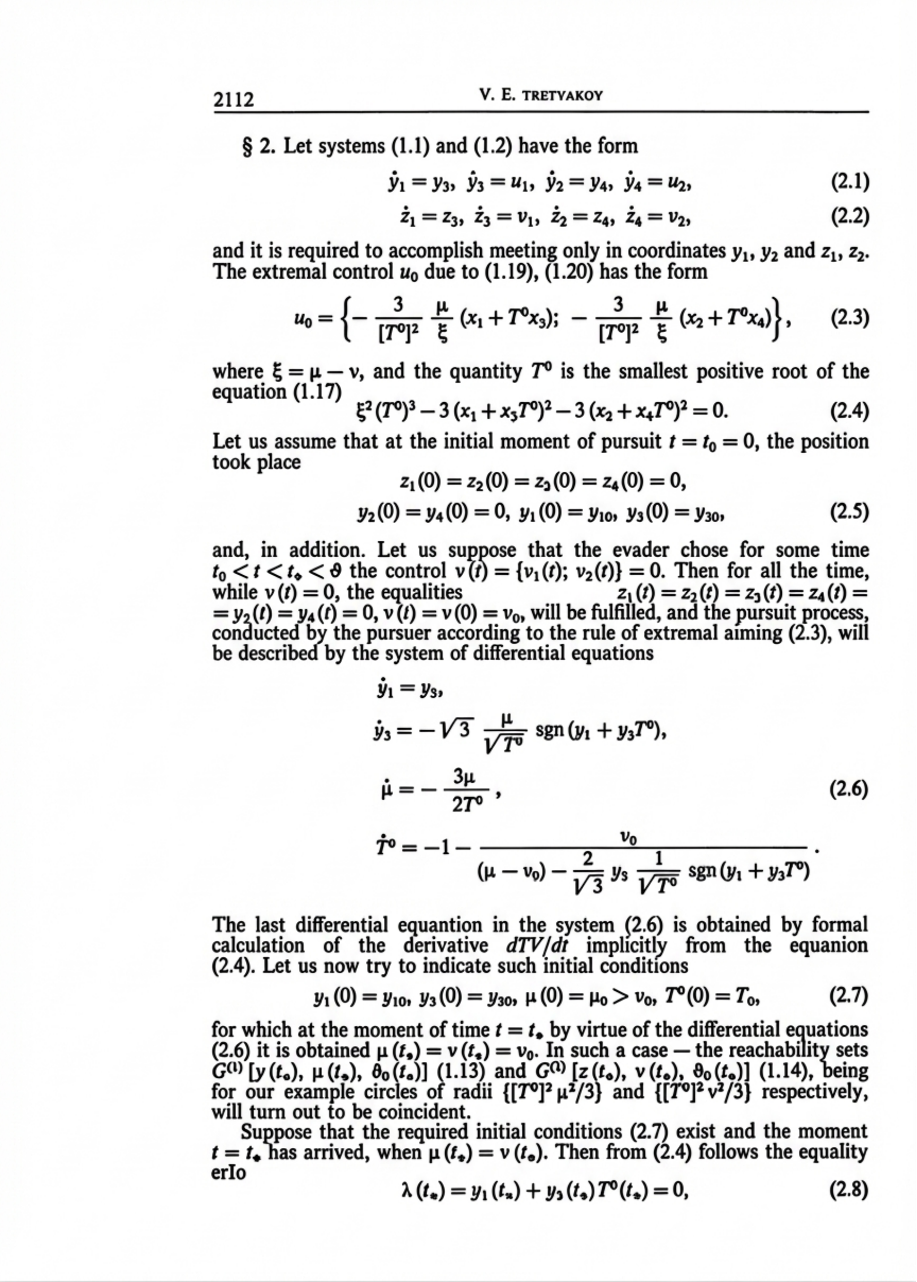
and under the condition

$$T^0 = \vartheta^0 - \tau = \min_{\omega} T. \tag{1.24}$$

In particular, it is also obtained that the moment of absorption ϑ_0 determined from (1.17) coiinades with the moment ϑ^0 of arrival of the movement $x(t)$ into position $x_{[m]} = 0$.

In paper [3], where the question was about meeting over all phase coordinates $(m = n)$, it was established that the extremal controls u_0 (1.19) and v_0 (1.21) are optimal strategies, solving problem 1 about pursuit. This circumsstance circumstance had a place due to the fact that with $u = u_0$ (1.19) and with any admissible v all the way up to the meeting, a situation could not arise where the boundaries of the reachability regions $G^{(1)}$ (1.13) and $G^{(2)}$ (1.14) touch in more than one mon., a point, if only at the initial moment of pursuit $t = t_0$ the indicated touching occurred in a single point $q^0(t_0)$ (1.16). The situation is more complex in the considered case $m < n$. As follows from the subsequent discussion, here the extremal control u_0 (1.19) no longer guarantees meeting of the movements $y(t)$ and $z(t)$ for any admissible control v during the time $\tau < t < \tau + T^0(\tau)$, and the rule of extremal aiming does not ensure $\min_u \max_v T = \min_u \cdot \max_v T = T^0$. In other words, the extremal controls $u = u_0$ (1.19) and $v = v_0$ (1.21) do not form a pair of optimal strategies for $m < n$. This assertion is proved by the following example.

Figure 4: Figure 4



§ 2. Let systems (1.1) and (1.2) have the form

$$\dot{y}_1 = y_3,\ \dot{y}_3 = u_1,\ \dot{y}_2 = y_4,\ \dot{y}_4 = u_2, \tag{2.1}$$

$$\dot{z}_1 = z_3,\ \dot{z}_3 = v_1,\ \dot{z}_2 = z_4,\ \dot{z}_4 = v_2, \tag{2.2}$$

and it is required to accomplish meeting only in coordinates y_1, y_2 and z_1, z_2. The extremal control u_0 due to (1.19), (1.20) has the form

$$u_0 = \left\{-\frac{3}{[T^0]^2}\frac{\mu}{\xi}(x_1 + T^0 x_3);\ -\frac{3}{[T^0]^2}\frac{\mu}{\xi}(x_2 + T^0 x_4)\right\}, \tag{2.3}$$

where $\xi = \mu - \nu$, and the quantity T^0 is the smallest positive root of the equation (1.17)

$$\xi^2 (T^0)^3 - 3(x_1 + x_3 T^0)^2 - 3(x_2 + x_4 T^0)^2 = 0. \tag{2.4}$$

Let us assume that at the initial moment of pursuit $t = t_0 = 0$, the position took place

$$z_1(0) = z_2(0) = z_3(0) = z_4(0) = 0,$$
$$y_2(0) = y_4(0) = 0,\ y_1(0) = y_{10},\ y_3(0) = y_{30}, \tag{2.5}$$

and, in addition. Let us suppose that the evader chose for some time $t_0 < t < t_* < \vartheta$ the control $v(t) = \{v_1(t);\ v_2(t)\} = 0$. Then for all the time, while $v(t) = 0$, the equalities $z_1(t) = z_2(t) = z_3(t) = z_4(t) = y_2(t) = y_4(t) = 0$, $\nu(t) = \nu(0) = \nu_0$, will be fulfilled, and the pursuit process, conducted by the pursuer according to the rule of extremal aiming (2.3), will be described by the system of differential equations

$$\begin{aligned} \dot{y}_1 &= y_3, \\ \dot{y}_3 &= -\sqrt{3}\,\frac{\mu}{\sqrt{T^0}}\,\operatorname{sgn}(y_1 + y_3 T^0), \\ \dot{\mu} &= -\frac{3\mu}{2T^0}, \\ \dot{T}^0 &= -1 - \frac{\nu_0}{(\mu - \nu_0) - \dfrac{2}{\sqrt{3}}\,y_3\,\dfrac{1}{\sqrt{T^0}}\,\operatorname{sgn}(y_1 + y_3 T^0)}. \end{aligned} \tag{2.6}$$

The last differential equantion in the system (2.6) is obtained by formal calculation of the derivative *dTV/dt* implicitly from the equanion (2.4). Let us now try to indicate such initial conditions

$$y_1(0) = y_{10},\ y_3(0) = y_{30},\ \mu(0) = \mu_0 > \nu_0,\ T^0(0) = T_0, \tag{2.7}$$

for which at the moment of time $t = t_*$ by virtue of the differential equations (2.6) it is obtained $\mu(t_*) = \nu(t_*) = \nu_0$. In such a case — the reachability sets $G^{(1)}[y(t_*),\ \mu(t_*),\ \vartheta_0(t_*)]$ (1.13) and $G^{(1)}[z(t_*),\ \nu(t_*),\ \vartheta_0(t_*)]$ (1.14), being for our example circles of radii $\{[T^0]^2\mu^2/3\}$ and $\{[T^0]^2\nu^2/3\}$ respectively, will turn out to be coincident.

Suppose that the required initial conditions (2.7) exist and the moment $t = t_*$ has arrived, when $\mu(t_*) = \nu(t_*)$. Then from (2.4) follows the equality erIo

$$\lambda(t_*) = y_1(t_*) + y_3(t_*)T^0(t_*) = 0, \tag{2.8}$$

Figure 5: Figure 5