ON THE STABILITY OF BRANCHING ALMOST-PERIODIC SOLUTIONS OF CERTAIN SYSTEMS OF DIFFERENTIAL EQUATIONS

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Abstract

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ON THE STABILITY OF BRANCHING ALMOST-PERIODIC SOLUTIONS OF CERTAIN SYSTEMS OF DIFFERENTIAL EQUATIONS

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1. Consider a system of ordinary differential equations

$$dx/dt = f(t, x; \mu) \qquad (f(t, 0; 0) \equiv 0). \tag{1}$$

Here μ is a scalar parameter, and x is a vector of the n-dimensional space E_n. We shall assume that the vector function $f(t, x; \mu)$ is almost periodic in t for fixed x, μ, and analytic in the space variables and in μ in some neighborhood T of the point $(0, 0)$, uniformly with respect to $t \in (-\infty, +\infty)$. It follows from these assumptions that the vector function $f(t, x; \mu)$ is almost periodic in t uniformly with respect to $x, \mu \in T$, i.e., it can be expanded in a Fourier series:

$$f(t, x; \mu) \sim \sum_{\nu=0}^{\infty} a_\nu(x, \mu) e^{i\lambda_\nu t}, \tag{2}$$

where the exponents λ_ν do not depend on x, μ.

Obviously, for $\mu = 0$ the system (1) has the trivial solution. If, in the representation (2), the numbers λ_ν are such that $\lambda_0 = 0,\ \lambda_\nu \geq \gamma > 0\ (\nu \neq 0)$, then, as was shown in (1), for small μ $(\mu \neq 0)$ the system (1) may have small nonzero almost-periodic solutions. The question of the existence and number of such solutions reduces to the analysis of a finite-dimensional system of branching equations.

In the present note, theorems will be given on the existence and number of branching small almost-periodic solutions of the system (1), and the question of their Lyapunov stability will be studied.

We note that analogous results in the periodic case were obtained in (2,3).

2. Denote by B_γ the class of almost-periodic vector functions whose Fourier exponents satisfy the conditions $\lambda_0 = 0,\ \lambda_\nu \geq \gamma > 0\ (\nu \neq 0)$. We shall describe the conditions imposed on the right-hand sides of the system (1). Suppose that, for fixed $x, \mu \in T$, the vector function $f(t, x; \mu)$ belongs to B_γ for some $\gamma > 0$. In T, the vector function $f(t, x; \mu)$ can be written in the form

$$f(t, x; \mu) = \sum_{i+j=1}^{\infty} C_{ij}[t; x]\mu^j, \tag{3}$$

where $C_{ij}[t; x]$ $(i \geq 1)$ are homogeneous polynomials of degree i in x. Suppose that

$$C_{20}[t; x] \equiv \ldots \equiv C_{k-1,0}[t; x] \equiv 0, \qquad C_{k0}[t; x] \not\equiv 0.$$

Let the matrix

$$V = C_{10}[t, \cdot]$$

not depend on t. If the matrix V_0 has no eigenvalues on the imaginary axis, then the system (1), for every small μ $(\mu \neq 0)$, has a unique-

almost-periodic solution $x(t, \mu)$ $(x(t, 0) \equiv 0)$, depending analytically on μ. Here we shall study the case when the matrix V_0 has on the imaginary axis one eigenvalue–zero.

3. Under the assumptions indicated above, let us consider the question of the existence and number of small almost-periodic solutions of system (1). Let one Jordan block correspond to the zero eigenvalue of the matrix V_0.* Let $e_0, e_1, \ldots, e_{m-1}$ be the eigenvector and associated vectors of the matrix V_0 corresponding to the zero eigenvalue, i.e. $V_0 e_0 = 0,\ V_0 e_i = e_{i-1}$ $(i = 1, \ldots, m-1)$, and let the equation $V_0 \xi = e_{m-1}$ have no solutions. Let $g_0, g_1, \ldots, g_{m-1}$ be a biorthogonal system $((e_i, g_i) = 1$ for $i + j = m-1,\ (e_i, g_i) = 0$ for $i + j \neq m-1)$ of the eigenvector and associated vectors of the matrix V_0^*, adjoint to V_0, also corresponding to the zero eigenvalue.

Introduce into consideration the numbers

$$M = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} (C_{k0}[s, e_0],\, g_0)\, ds, \tag{4}$$

$$N = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} (C_{0,1}(s),\, g_0)\, ds, \tag{5}$$

where (x, y) is the ordinary scalar product of the vectors $x, y \in E_n$.

Theorem 1. Let $NM \neq 0$.

Then system (1), for each sufficiently small μ ($\mu \neq 0$), has k small almost-periodic solutions $x_j(t, \mu)$ ($x_j(t, 0) \equiv 0,\ j = 1, 2, \dots, k$), belonging to the space B_γ.

If $M \neq 0,\ N = 0$, then one can give formulas for determining such sets of numbers that the nonvanishing of each member of this set makes it possible to judge the number of small almost-periodic solutions of system (1). The case $M = 0$ is special and requires additional consideration.

4. We now turn to the consideration of the question of the stability of the small almost-periodic solutions of system (1), whose existence is guaranteed by Theorem 1. For this purpose let us additionally assume that, for each sufficiently small μ, all solutions of system (1) which at $t = 0$ take values from some small ball can be continued to the whole half-axis $[0, +\infty)$.

Let us note at once that the presence of at least one eigenvalue of the matrix V_0 inside the right half-plane entails Lyapunov instability of all the emerging small almost-periodic solutions of system (1) belonging to B_γ. Therefore, in considering the question of the stability of small almost-periodic solutions, only the case in which the matrix has no eigenvalues inside the right half-plane is of interest.

Theorem 1 can be supplemented by the following statements.

Theorem 2. Let $m \geq 3,\ NM \neq 0$.

Then all emerging small almost-periodic solutions $x_j(t, \mu)$ ($x_j(t, 0) = 0,\ j = 1, 2, \dots, k$) of system (1), belonging to the class B_γ, are Lyapunov unstable.

For $m \leq 2$, both stable and unstable almost-periodic solutions may emerge. To formulate the corresponding results, introduce the following notation.

The roots of the equation

$$z^k + MN^{k-1} = 0 \tag{6}$$

will be denoted by $z_1, \dots, z_k$. Let s be the number of those roots z_i of equation (6) which satisfy the condition $\operatorname{Re} \sqrt{z_i} \neq 0$.

* The case when several Jordan blocks correspond to the zero eigenvalue of the matrix is considered analogously, but the calculations become more complicated.

Theorem 3. Let $m = 1,\ k = 2l,\ NM \neq 0$.
Then, for every small μ, exactly l small almost-periodic solutions of system (1) belonging to the class B_γ are born that are asymptotically stable in the sense of Lyapunov, and exactly l are unstable in the sense of Lyapunov.

If $m = 1$ and k is an odd number, then for every small μ the number of asymptotically stable small solutions is equal to the number of roots of equation (6) lying in the left half-plane, and the number of unstable solutions is equal to the number of roots lying in the right half-plane.

Theorem 4. Let $m = 2,\ NM \neq 0$.
Then, for every small μ, s small almost-periodic solutions belonging to the class B_γ are born, unstable in the sense of Lyapunov.

5. In the proof of Theorems 2–4 we used the basic facts of the perturbation theory of linear operators (see, for example, (4)) and Lillo' s results (5) on the reducibility of certain classes of systems of linear differential equations with almost-periodic coefficients.

The general scheme for applying methods of the perturbation theory of linear operators to questions of stability of periodic solutions of systems of ordinary differential equations was indicated by M. A. Krasnosel' skii (6). According to this scheme, the problem of stability reduces to the study of perturbations of the multipliers (eigenvalues of the shift operator) linearized on a periodic solution of the system. In the almost-periodic case considered by us, this problem reduces to the study of perturbations of the eigenvalues of a matrix which is the mean value of the right-hand side linearized on an almost-periodic solution of the system.

6. Suppose that the matrix $C_{1,0}[t,\cdot]$ is not constant. Then we introduce into consideration the matrix

$$V_0 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} C_{1,0}[s,\cdot]\, ds.$$

Let $e_0,\ g_0$ be eigenvectors of the matrices $V_0,\ V_0^*$, corresponding to the zero eigenvalue. Denote by $x_0(t)$ the solution of the linear system $dx/dt = C_{1,0}[t;x]$ with the initial condition $x_0(0) = e_0$. Denote by $y_0(t)$ the solution of the adjoint system

$$dy/dt = -C_{1,0}^*[t,y]$$

with the initial condition $y(0) = y_0$.

All the considerations of the preceding items are valid, with the sole difference that the numbers $M,\ N$ are replaced by the numbers

$$M' = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} (C_{k,0}[s, x_0(s)],\ y_0(s))\, ds,$$

$$N' = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} (C_{0,1}(s),\ y_0(s))\, ds.$$

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Note: Figure translations are in progress. See original paper for figures.