SOME ESTIMATES FOR THE DERIVATIVE OF THE SOLUTION OF THE DIRICHLET PROBLEM ON THE BOUNDARY

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Abstract

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SOME ESTIMATES FOR THE DERIVATIVE OF THE SOLUTION OF THE DIRICHLET PROBLEM ON THE BOUNDARY

1. We consider solutions $u(x)$ of equations or inequalities of second order in an n-dimensional domain G. It is assumed that u is continuous in $G + \partial G$ and has in G an absolutely continuous supporting representation (^{1a}, ^{6}). We shall estimate the upper normal derivative u_ν on ∂G. Estimates for the lower derivative are obtained by obvious changes of signs. The general conditions on the equation and the method will be indicated at the end of the note, and for now let us assume that u satisfies the equation

$$a^{ij}u_{ij} + b^i u_i = g, \qquad a^{ij}\xi_i\xi_j \geqslant 0, \tag{1}$$

and the condition $u|_{\partial G} = 0$. It is sufficient that (1) hold at almost all points of convexity of u; if a^{ij}, b^i, g depend on u, then $u = u(x)$ is substituted into them; the derivatives u_i, u_{ij} may be understood as the coefficients of approximating differentials. The question of estimates of u_ν is far from new, but it has been considered under other conditions and, correspondingly, the results were of a different character.

Let ν be a unit vector and P a supporting plane to ∂G with outward normal ν. Let S be the graph of the function u in $(n+1)$-dimensional space. Through P pass n-dimensional supporting planes to S. Let $p(\nu)$ be the tangent of the angle of inclination of the extreme one of them from below. It is not excluded that $p(\nu) = \infty$. Defining the upper normal derivative at the point $x_0 \in P \cap \partial G$ by

$$u_\nu(x_0) = \limsup_{x \to x_0} \frac{u(x) - u(x_0)}{(x - x_0)\nu}, \tag{2}$$

we have, obviously, $p(\nu) \geqslant u_\nu(x)$ (incidentally, since $u|_{\partial G} = 0$, $u(x_0) = 0$).

For functions in G we introduce the norm

$$\|\varphi\| = \|a^{-1/n}\varphi\|_{L_n(G)}, \qquad a = \det(a^{ij}). \tag{3}$$

Denote by b the vector $(b^1, \ldots, b^n)$ and set $\|b\| = \||b|\|$.

Theorem 1. *For the geometric mean of the quantity $p(\nu)$ the estimate holds*

$$\exp \frac{1}{\varkappa_n} \int_\Omega \ln p(\nu)\, d\nu \leqslant n^{-1}\tau_n^{-1/n}\|g_+\|F_n(\|b\|), \tag{4}$$

where Ω is the unit sphere (the set of all ν); $\varkappa_n$ is its area, $\tau_n = n^{-1}\varkappa_n$, and the function

$$F_n(\xi) = e^{\xi^n/nn\varkappa_n + \varphi_n(\xi)}, \tag{5}$$

φ_n is bounded and $\varphi_n(0) = 0$; $\varphi_1(\xi) \equiv 0$.

The same function F_n enters into the estimates for $u(x)$ obtained in ([16]); there its exact definition is given. It is of importance in connection with the following theorem, asserting the sharpness of estimate (4):

Theorem 2. *In a ball, for arbitrary $B \geqslant 0$, $F \geqslant 0$, one can prescribe an equation (1) with $\|b\| = B$, $\|g\| = F$, having a solution with $u|_{\partial G} = 0$, for which equality holds in (4). In an arbitrary convex domain*

one can ensure, under the same conditions, that the left-hand side of (4) differs from the right-hand side by less than a given $\varepsilon > 0$, if B is taken sufficiently large. (In both cases one can ensure that the solution, the coefficients, and the right-hand side of the equation are arbitrarily smooth inside the domain and that $a^{ij}\xi_i\xi_j \geq \alpha(x)|\xi|^2$, $\alpha(x) > 0$, $u_\nu = p(\nu)$.)

2. Let $r(x)$ be the distance from x to the boundary of the convex hull of the domain G in the direction opposite to the vector $b(x)$.

Theorem 3. For the mean n-th power of the quantity $p(\nu)$ the inequality

$$\left[\frac{1}{\varkappa_n}\int_\Omega p^n(\nu)\, d\nu\right]^{1/n} \leq n^{-1}\tau_n^{-1/n}\|\bar{g}_+\|, \tag{6}$$

holds, where $\bar{g} = g - |b|r^{-1}u$. This inequality is sharp: in a ball, for a given $F \geq 0$, one can prescribe equation (1) with $b = 0$, $\|g\| = F$, having a solution with $u|_{\partial G} = 0$, for which equality holds in (6), while in any convex domain one can ensure that the left-hand side differs from the right-hand side by less than a given $\varepsilon > 0$. (Here one can ensure the same additional requirements as in Theorem 2.)

The requirement that the norm entering (6) be finite is rather strong, since it is possible that $r(x) \to 0$ as $x \to \partial G$. It is certainly fulfilled if $\|br^{-s}\| < \infty$ for some $s > (n-1)/n$. If, however, the upper radii of curvature of the boundary of the convex hull are bounded, then it is enough that $\|br^{-s}\| < \infty$ for some $s > (n-1)/2n$.

The estimates (3), (6) are of interest not only because of their sharpness, but also because they cannot be replaced by estimates with "weaker" norms. Namely, the following holds.

Theorem 4. Let $\varphi(\xi)$ be such a function on the half-axis, $\xi \geq 0$, that as $\xi \to \infty$, $\xi^{-1}\varphi(\xi) \to 0$. In the ball G one can prescribe equation (1) with $b = 0$ and

$$\int_G \varphi(a^{-1}|g|^n)\,dx < \infty,$$

having a solution with $u|_{\partial G} = 0$, such that everywhere on ∂G, $p(\nu) = u_\nu = \infty$.

3. Generalization. Let E be an m-dimensional plane, $m \geq 1$; x_E the projection of the point x onto E; G_E the projection of G; $a_E = \det(a^{ij})$, $i, j \leq m$, if by a rotation of the axes E has been made parallel to the plane $(x^1, \dots, x^m)$. We introduce, as in (1^6), the norm $\|\varphi\|'_E$ of a function in G. We consider measurable functions ψ in G_E such that in G

$$|\varphi(x)| \leq a_E^{1/m}(x)\psi(x_E),$$

and put

$$\|\varphi\|'_E = \inf \|\psi\|_{L_m(G_E)}. \tag{7}$$

By a pencil we shall mean a set of planes E passing through some $(m-1)$-dimensional plane E_0. We say that a pencil is not rarefied at the plane E if E passes through the same E_0 and in an arbitrarily small neighborhood of it there is contained a part of it of positive measure in the sense of the natural measure in the set of all E passing through E_0.

We define the norm $\|\varphi\|_E$ by setting

$$\|\varphi\|_E = \inf \sup_{\{E'\}} \|\varphi\|_{E'}, \tag{8}$$

the infimum of the quantities $\sup \|\varphi\|_{E'}$ over all pencils $\{E'\}$ not rarefied at E.

We introduce still more notation: b_E is the projection of the vector b onto the plane E; $r_E(x)$ is the distance from x_E to the boundary of the convex hull of the domain G_E in the direction opposite to $b_E(x)$.

Theorem 5. For every plane E the inequalities, entirely analogous to (3), (6), are valid:

$$\exp \frac{1}{\varkappa_m} \int_{\Omega_E} \ln p(\nu)\,d\nu \leq m^{-1}\tau^{-1/m}\|g_+\|_E F_m(\|b\|_E), \tag{9}$$

$$\left[\frac{1}{\varkappa_m}\int_{\Omega_E} p^m(\nu)\,d\nu\right]^{1/m} \leq m^{-1}\tau_m^{-1/m}\|\bar{g}_+\|_E, \qquad \bar{g} = g - |b_E| r_E^{-1} u, \tag{10}$$

where Ω_E is the unit sphere in E. (Formally, the value ∞ is allowed for norms.) The same inequalities with norms (7) are valid for almost all E in any pencil.

If, however, $u(x)$ is differentiable at every point of convexity in some direction and, at all the same points, except perhaps for a countable set, u_i, u_{ij} exist and (1) is satisfied, then (9), (10) are valid with norms (7) for all planes without exception.

There is also an assertion on the sharpness of the estimates (9), (10), analogous to Theorem 2 and the corresponding part of Theorem 3. The only difference is that now the equation must be prescribed in the hypercylinder $G_E \times G_{E'}$, where G_E is a ball or, respectively, an arbitrary convex domain in E, and $G_{E'}$ is an arbitrary domain in the $(n-m)$-dimensional plane completely perpendicular to E.

4. The simplest case is when $m = 1$, i.e. E are straight lines. Then let us denote by ν the unit vector along E and replace the index E by ν; in this case $b_E = b\nu$. Then (9), (10) give

$$\sqrt{p(\nu)p(-\nu)} \leq \tfrac{1}{2}\|g_+\|_\nu e^{\|b\|_\nu/2}, \tag{11}$$

$$p(\nu) + p(-\nu) \leq \|\bar{g}_+\|_\nu. \tag{12}$$

The following theorem gives an estimate for $p(\nu)$ separately.

Theorem 6. For every ν

$$p(\nu) \leq \|g_+\|_\nu e^{\|(b\nu)_-\|_\nu}, \tag{13}$$

where $(b\nu)_-$ is the negative part of $b\nu$, and the same estimate with norm (7) is valid for almost all ν, or for all ν under the conditions of the latter part of Theorem 5. This estimate is sharp in the sense indicated in Theorem 5.

5. If at a point $x \in \partial G$ there is no supporting plane, then by a suitable transformation of the domain one can make one exist.

Then, if for the transformed equation and solution the conditions of Theorem 6 are satisfied, we obtain from it an estimate of $u_\nu(x)$ for the original solution. In this way we obtain, for example, the following result.

Let the solution u of equation (1) have generalized derivatives u_{ij}, locally summable with the n-th power, and let, for $x_0 \in \partial G$ and its r-neighborhood U, the following conditions be satisfied.

(I) If x_0 is taken as the origin, then, for a suitable choice of axes in $U \cap G$,

$$x_1 < A\left(\sum_2^n x_i^2\right)^{1/2},$$

where $A > 0,\ l > 1$. Figuratively speaking, the paraboloid

$$x_1 = A\left(\sum x_i^2\right)^{1/2}$$

touches $U \cap G$ from outside at the point x_0.

(II) In $U \cap G$, $|a^{ij}|,\ |b|,\ |g| \leq M$ and $a^{ij}\xi_i\xi_j \geq a|\xi|^2,\ a = \text{const} > 0$.

Theorem 7. Under the indicated conditions, in $U \cap G$

$$|u(x)| \leq N|x - x_0|, \qquad N = N(n, a^{-1}M, r, l, A). \tag{14}$$

Corollary. If the same conditions on $u(x)$ and equation (1) are fulfilled in all of G and $\partial G \in C^{1,\alpha}$, then the upper and lower normal derivatives on ∂G are bounded in terms only of n, $a^{-1}M$, the exponent α and the Hölder coefficient for the normals to ∂G, and the diameter of G.

The reduction of Theorem 7 to Theorem 6 is carried out by the transformation

$$y_1 = x_1 - 2A\left(\sum_2^n x_i^2\right)^{1/2}, \qquad y_2 = x_2, \dots,\ y_n = x_n.$$

Condition (II) ensures finiteness of $\|b\|_\nu$, $\|g\|_\nu$ for the transformed equation. The same is ensured under weaker conditions as well, so that from Theorem 6 it is easy to derive stronger results too. If, however, the whole domain G is transformed into a convex one by a sufficiently smooth transformation, then Theorems 1-6 imply the corresponding estimates for the normal derivative on all of ∂G.

6. The estimates given in §§ 1-4 are applications of the general method used in (^{1a,b,c}) for estimating $u(x)$. Suppose that, under the assumptions of § 1, the following inequality is satisfied at almost all points of convexity:

$$F(u_{ij}, u_i, u, x) \leqslant 0. \tag{15}$$

Suppose it follows from this that, at almost all the same points,

$$w \leqslant X(x)U(\nabla u), \qquad w = \det(u_{ij}), \tag{16}$$

where it is assumed that $X \geqslant 0$ is defined throughout G, and in deriving (16) substitutions $u = u(x), \dots$ are allowed. As shown in (^{1a}), the derivation of (16) from (15) is possible for a very broad class of functions F; in a certain sense, for all elliptic F. In particular, if $a^{ij}\xi_i\xi_j \geqslant 0$, $d^2u \geqslant 0$, then $a^{ij}u_{ij} \geqslant n(aw)^{1/n}$ ($a = \det(a^{ij})$), and therefore from (1) there follows an inequality of the form (16).

Theorem 8. *Under the stated assumptions the following inequality holds:*

$$\int_\Omega \left(\int_0^{p(\nu)} U^{-1}(p^\nu) p^{n-1}\, dp \right) d\nu \leqslant \int_G X(x)\, dx, \tag{17}$$

where $p = |\nabla u|$, $\nabla u = p\nu$, and $p(\nu)$ is the same as in § 1.

Proof. From (16), $U^{-1}w \leqslant X$. Integrating over the set of points of convexity of u and observing that w is the Jacobian of the transformation $(x_i) \to (u_i)$, we obtain on the left the integral of $U^{-1}du_1 \dots du_n$ (since the support mapping is absolutely continuous!). Passing in this integral to the variables p, ν, we obtain (17). It is clear that (17) means an estimate of a certain mean value of $p(\nu)$. The estimates of Theorems 1 and 2 are its special cases corresponding to equation (1); their derivation is analogous to the derivation of the estimates of $|u(x)|$ in (^{1b}). The method applied in (^{1a,b}) leads to a generalization of Theorem 8, giving an estimate of the means of $p(\nu)$ on the spheres Ω_E in different planes E. The estimates of Theorem 5 turn out to be their special cases; their derivation is analogous to the derivation of the estimates of $|u|$ and (^{1b}).

Theorem 6 follows directly from Theorem 2 of the note (^{1c}), where one general method is given for deriving estimates of $p(\nu)$ at points of convexity of ∂G.

The assertions on the sharpness of our estimates and Theorem 4 are proved similarly to how the corresponding assertions were proved in (^{1g,d}) for estimates of $|u|$.

The condition $u|_{\partial G} = 0$ can be removed, for example, in the same way as was done in (^{1a,b}). If v is the greatest of the convex functions with $v|_{\partial G} \leqslant u|_{\partial G}$, then for $\bar{u} = u - v$, $\bar{u}|_{\partial G} = 0$ at all points of ∂G through which support planes pass, except perhaps those lying inside the flat faces of the convex hull of G (and these correspond to at most a countable set of normals ν). The points of convexity of $\bar{u}$ correspond to the points of convexity of u, and at them $\bar{w} \leqslant w$. Estimating $\bar{u}_\nu$, we obtain an estimate for $u_\nu = \bar{u}_\nu + v_\nu$, when v_ν is estimated from the properties of the boundary values $u|_{\partial G}$.

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