INTEGRAL INEQUALITIES WITH OPERATORS OF VOLTERRA TYPE

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Abstract

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By E^+ we shall denote a linear partially ordered set of nonnegative elements $u, v, \psi, \ldots$, and by K^+ the set of operators L, l, L_i $(i = 1, \ldots, n)$ of Volterra type that map the set E^+ into itself, for which the series

$$R_i(\lambda_i) = (I - \lambda_i L_i)^{-1} = \sum_{k=0}^{\infty} \lambda_i^k L_i^k$$

converge for every value of the complex parameter λ_i. We shall assume that the set K^+, together with any two operators, contains their sum, product, and the operator $T_i(\lambda) = L_i R_i(\lambda)$, if $L_i \in K^+$ $(\lambda \geq 0)$.

We shall be interested in the following properties of the operators. Multiplying the equality

$$L_i = (\lambda - \alpha)^{-1}[(I - \alpha L_i) - (I - \lambda L_i)] \qquad (\lambda \neq \alpha)$$

by $R_i(\alpha) R_i(\lambda)$, we easily arrive at the relation

$$T_i(\alpha) R_i(\lambda) = (\lambda - \alpha)^{-1}[R_i(\lambda) - R_i(\alpha)],$$

whence, for $\lambda > \alpha \geq 0$, in turn follows the inequality

$$T_i(\alpha) R_i(\lambda) \leq (\lambda - \alpha)^{-1}[R_i(\lambda) - I] < (\lambda - \alpha)^{-1} R_i(\lambda).$$

Using this inequality, by the method of complete mathematical induction one can show that

$$T_i^m(\alpha) R_i(\lambda) < (\lambda - \alpha)^{-m}[R_i(\lambda) - I]. \tag{1}$$

Let now the operators L_i satisfy the relation $L_{iL}k \leq L_{kL}i$ for $i > k$.

Then, if $\alpha_i \geq 0$, $\alpha_k \geq 0$, $i > k$, then

$$T_i(\alpha_i)R_k(\alpha_k) \leq R_k(\alpha_k)T_i(\alpha_i). \tag{2}$$

Consider the operator $R = R_1(\lambda_1) \cdots R_n(\lambda_n)$. If $\lambda_i > \alpha_i \geq 0$, using inequalities (1) and (2), one can obtain that

$$T_i^m R \leq (\lambda_i - \alpha_i)^{-m}(R - I). \tag{3}$$

In what follows we shall assume that the nonnegative real numbers λ_i and α_{ij} $(i = 1, \dots, n;\ j = 1, \dots, p)$ satisfy the inequality $\lambda_i > \alpha_{ij}$, and m_{ij} will denote integers.

For simplicity put $T_{ij} = T_i(\alpha_{ij})$. Starting from inequality (3), by the method of complete mathematical induction one can finally prove the inequality

$$\prod_{i=1}^{n}\prod_{j=1}^{p} T_{ij}^{m_{ij}} R \leq \prod_{i=1}^{n}\prod_{j=1}^{p}(\lambda_i - \alpha_{ij})^{-m_{ij}}(R - I), \tag{4}$$

Let us introduce into consideration the operator

$$L = \sum_{m=1}^{N} a(m_{11}, \dots, m_{ij}, \dots, m_{np}) \prod_{i=1}^{n}\prod_{j=1}^{p} T_{ij}^{m_{ij}},$$

where, respectively, $m = \sum_{i=1}^{n}\sum_{j=1}^{p} m_{ij}$ are natural numbers, and the coefficients a are nonnegative numbers. The equation

$$F(\lambda) \equiv \sum_{m=1}^{N} a(m_{11}, \dots, m_{np}) \prod_{i=1}^{n}\prod_{j=1}^{p}(\lambda_i - a_{ij})^{-m_{ij}} = 1, \tag{5}$$

where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$, will be called characteristic for the equation $u = \psi + Lu$. Obviously, $L \in K^+$.

Let us formulate the main inequality.

Theorem 1. If an element u satisfies the inequality $u \leqslant \psi + Lu$ $(u, \psi \in E^+)$, then

$$u \leqslant R\psi = R_1(\lambda_1) \cdots R_n(\lambda_n)\psi \equiv v(\lambda), \tag{6}$$

where $\lambda_1, \dots, \lambda_n$ satisfy the characteristic equation (5).

To prove the assertion of the theorem, consider the expression $\psi + Lv$. From inequality (4), taking into account that λ satisfy relation (5), it is easy to obtain the inequalities

$$\psi + Lv = \psi + LR\psi \leqslant \psi + F(\lambda)(R - I)\psi = \psi + (R - I)\psi = R\psi = v,$$

i.e., $v \geqslant \psi + Lv$. Now the assertion of the theorem follows from the following proposition.

Lemma. If u, v satisfy the inequalities $u \leqslant \psi + Lu,\ v \geqslant \psi + Lv$, where $L \in K^+$, then $u \leqslant v$.

The assertion of the lemma is proved as follows: subtracting the first inequality from the second, we obtain $v - u \geqslant L(u - v)$. Hence $v - u = \psi_1 + L(v - u)$ $(\psi_1 \in E^+)$, and $v - u = (I - L)^{-1}\psi_1 \geqslant 0$.

Remark 1. The characteristic equation (5) for $n > 1$ has an infinite set of solutions. To each solution there corresponds its own estimate of the form (6). Denote $v_* = \inf_{F(\lambda)=1} v(\lambda)$. It can be proved: if u satisfies the condition of Theorem 1, then $u \leqslant v_*$.

For operators of a special form one can also obtain the reverse inequality.

Theorem 2. If v satisfies the inequality $v \geqslant \psi + \sum_{k=1}^{n} a_k L_k v$, then

$$v \geqslant R_n(a_n) \cdots R_1(a_1)\psi.$$

Corollary. From Theorems 1 and 2 there follows the assertion: if the operator l is defined by the rule

$$lu = \sum_{k=1}^{n} a_k L_k u,$$

then

$$R_n(a_n) \cdots R_1(a_1) \leqslant (I - l)^{-1} \leqslant R_1(\lambda_1) \cdots R_n(\lambda_n),$$

where the nonnegative numbers $\lambda_1, \dots, \lambda_n$ satisfy the relation

$$\lambda_1^{-1} a_1 + \cdots + \lambda_n^{-1} a_n = 1. \tag{7}$$

Examples. If

$$L_k u = \int_0^{t_k} \alpha_k(s_k)\, u(t_1, \dots, t_{k-1}, \tau_k, t_{k+1}, \dots, t_n)\, d\tau_k, \tag{8}$$

then

$$T_k(\lambda_k)\psi = \int_0^{t_k} \alpha_k \exp\left\{\lambda_k \int_{s_k}^{t_k} \alpha_k \, d\tau_k\right\} \psi(\dots, t_{k-1}, s_k, t_{k+1}, \dots) \, ds_k. \quad (9)$$

and $R_k(\lambda_k) = I + \lambda_k T_k(\lambda_k)$, where the operators L_k $(k = 1, \dots, n)$ commute pairwise.

From Theorems 1 and 2 one can derive the following inequalities.

1. If the function $u(t)$ satisfies the inequality

$$u(t) \leq \psi(t) \int_0^t +\alpha u \, d\tau,$$

then

$$u(t) \leq \psi(t) + \int_0^t \alpha \exp\left\{\int_s^t \alpha \, d\tau\right\} \psi \, d\tau.$$

This is the well-known Gronwall–Bellman inequality ([1]).

2. If the function $u = u(t_1, \dots, t_n)$ satisfies the inequality

$$u \leq \psi + \sum_{k=1}^{n} L_k u,$$

then

$$u \leq \prod_{k=1}^{n} (I + \lambda_k T_k)\psi = \psi + \sum_{k=1}^{n} \lambda_k T_k \psi + \dots + \lambda_1 \cdots \lambda_n T_1 \cdots T_n \psi;$$

in particular, if $\psi = C > 0$ is constant, then

$$u \leq C \exp\left\{\sum_{k=1}^{n} \lambda_k \int_0^{t_k} \alpha_k \, d\tau_k\right\}; \quad (10)$$

if

$$\psi = \int_0^{t_1} \cdots \int_0^{t_n} \varphi \, d\tau_n \dots d\tau_1,$$

then

$$u \leq \int_0^{t_1} \cdots \int_0^{t_n} \exp\left\{\sum_{k=1}^{n} \lambda_k \int_{s_k}^{t_k} \alpha_k \, d\tau_k\right\} \varphi \, d\tau_n \dots d\tau_1, \quad (11)$$

where the operators $L_k, T_k = T_k(\lambda_k)$ are determined from relations (8) and (9), $\psi \geq 0$, $\lambda_1^{-1} + \dots + \lambda_n^{-1} = 1$.

These inequalities improve some inequalities of Wendroff ([1]) and inequalities from ([2]).

Remark. By varying the numbers λ_k, instead of inequalities (10) and (11) one can obtain the sharper inequalities

$$u \leq C \exp \left\{ \sum_{k=1}^{n} \left[\int_0^{t_k} \alpha_k \, d\tau_k \right]^{1/2} \right\}^2 ,$$

$$u \leq \int_0^{t_1} \dots \int_0^{t_n} \exp \left\{ \sum_{k=1}^{n} \left[\int_{s_k}^{t_k} \alpha_k \, d\tau_k \right]^{1/2} \right\}^2 \varphi \, ds_n \dots ds_1 .$$

3. Let now

$$L_k u = \int_0^t e^{a_k(t-\tau)} u \, d\tau \qquad (k = 1, \dots, n).$$

Then the operators L_k commute among themselves and

$$R_k(\lambda_k)\psi = \psi + \lambda_k \int_0^t e^{(\lambda_k + a_k)(t-\tau)} \psi(\tau) \, d\tau.$$

One can also find constants C_k $(k = 1, \dots, n)$ such that

$$\prod_{k=1}^{n} R_k(\lambda_k)\psi \leq \psi + \sum_{k=1}^{n} C_k \int_0^t e^{(\lambda_k + a_k)(t-\tau)} \psi(\tau) \, d\tau. \tag{12}$$

Therefore, if the function $u(t)$ satisfies the inequality

$$u \leqslant \psi + \sum_{k=1}^{n} a_k \int_0^t e^{\alpha_k(t-\tau)} u \, d\tau,$$

where $a_k \geqslant 0$, $\psi \geqslant 0$, then the function u will be less than the right-hand side of relation (12) for a corresponding choice of the constants C_k. ($\lambda_k > 0$ are determined from relation (7).)

4. From Theorems 1 and 2 one can obtain various inequalities also with matrix coefficients; for example, if

$$u(t) \leqslant \psi(t) + A \int_0^t e^{P(t-\tau)} u \, d\tau + B \int_0^t e^{Q(t-\tau)} u \, d\tau,$$

then

$$u(t) \leqslant \psi(t) \lambda A \int_0^t e^{(\lambda A + P)(t-\tau)} \psi(\tau) \, d\tau + \mu B \int_0^t e^{(\mu B + Q)(t-\tau)} \psi(\tau) \, d\tau$$
$$+ \lambda\mu AB \int_0^t e^{(\lambda A + P)(t-s)} \int_0^s e^{(\mu B + Q)(s-\tau)} \psi(\tau) \, d\tau,$$

where u, ψ are nonnegative vector functions; A, B, P, Q are square matrices with nonnegative components that are pairwise permutable with one another; $\lambda > 1,\ \mu > 1$ satisfy the equality $\lambda^{-1} + \mu^{-1} = 1$.

These inequalities can be used to prove theorems on existence, uniqueness, continuous dependence, and stability of solutions of differential and integral equations in many variables.

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Note: Figure translations are in progress. See original paper for figures.