CONVOLUTION-TYPE OPERATORS IN CONES

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Abstract

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In the present paper the general method set forth in ([1,2]) is applied to multidimensional convolution operators. It gives, for example, necessary and sufficient Noetherian conditions for multidimensional Wiener–Hopf equations in smooth cones. An essential role here is played by an expansion of Euclidean space specially adapted to the given problems ([20]). In the paper new classes of operators are constructed (generalized convolutions, composite convolutions), for which necessary and sufficient Noetherian conditions are obtained. We note that the proposed method also provides the possibility of numerical solution by regularization followed by solution of Fredholm equations.

The study of convolution-type operators makes it possible to consider the associated boundary-value problems for analytic functions of many complex variables in tubular domains. In an analogous way one can also consider discrete convolutions and the boundary-value problems associated with them, for example, the problem of linear conjugation for bicylinders. We shall not touch upon this question here.

One-dimensional Wiener–Hopf equations and their discrete analogues have by now been considered with great completeness. Their solution was carried out by the method of factorization of the coefficients of the corresponding boundary-value problem ([3–8]). Exceptions are the works ([4,5]), whose analysis was one of the starting points of the theory expounded here. For many variables only the case of equations on a half-space has been completely investigated ([9,10]). In those same works (for example ([11,12])), which were devoted to the study of convolutions in other domains (cones, angles) and of the boundary-value problems associated with them, the authors also used the Wiener–Hopf method, i.e. the method of factorization. This forced one to impose on the kernel of the equation requirements connected exclusively with the method and therefore very far from necessary.

In this paper we proceed in the opposite direction, namely: we investigate convolutions in themselves, simultaneously obtaining information about the boundary-value problems associated with them (without resorting to factorization).

1°. **The ring w_p^n.** The space of vector-functions (matrix-functions) of dimension n, defined on the Euclidean space E_m and summable to the power p $(1 \le p < \infty)$, will be denoted by $\mathcal{L}_p^n(E_m)$ $(\mathcal{L}_p^{n,n}(E_m))$. It is known that operators K of the form

$$(Kf)(x) = cf(x) + \int_{E_m} k(x-y)f(y)\,dy, \tag{1}$$

where $k \in \mathcal{L}_1^{n,n}(E_m)$, are linear operators $\mathcal{L}_p^n \to \mathcal{L}_p^n$. We shall denote by w_p^n the closure, in the operator norm, of the set of operators of the form (1). It is clear that operators from w_p^n are invariant with respect to shifts. In ([13]) it was established that every operator A, invariant with respect to shifts, after the Fourier transform becomes an operator of multiplication by a measurable bounded matrix-function $\hat{A}(\xi)$, which, following ([14]), we shall call the symbol of the operator A. The space consisting of functions of the form $\hat{A}$, where $A \in w_p^n$, will be denoted by m_p^n. The norm in m_p^n is introduced by the equality $\|\hat{A}\|_{m_p^n} = \|A\|_p$. In the space m_2^1, obviously, there holds the equality

$$\|A\|_2 = \operatorname{ess\,max} |\hat{A}(\xi)|, \quad \xi \in \dot{E}_m.$$

It follows that the functions from m_p^n are continuous on $\dot{E}_m$, and the space m_2^n coincides (also topologically) with the space of matrix-functions continuous on $\dot{E}_m$. Here $\dot{E}_m$ denotes the Euclidean space compactified by one point at infinity.

2°. We carry out the compactification of E_m as follows. To each ray issuing from the origin we assign a point at infinity. On the union of the set of points of E_m and the points at infinity we introduce a topology. As a fundamental system of neighborhoods of each point at infinity x_∞ we take sets of the form $(M \cap \Gamma) \cup \Gamma_\infty$. Here Γ is any open cone containing the ray corresponding to the point x_∞; Γ_∞ is the collection of points at infinity corresponding to rays lying inside Γ; M is the exterior of an arbitrary closed disk. The space E_m compactified in this way is homeomorphic to the closed ball of dimension m. We shall denote the indicated compactification by $\tilde{E}_m$, and the collection of points at infinity by $\mathfrak{N}$. We extend the measure to the compactified space $\tilde{E}_m$ by declaring $\operatorname{mes} \mathfrak{N} = 0$.

In what follows, without further reservations we shall use the concepts and notation of ([1],[2]).

Lemma 1. *An operator $A(\in w_p^n)$ is an operator of local type.*

Lemma 2. *If two operators from w_p^n are locally equivalent at some point at infinity, then they are equal. If the point x is finite, then*

$$A \overset{x}{\sim} c_1 I \quad (I \text{ is the identity operator, } c_1 = \hat{A}(\infty)).$$

3°. An operator

$$A(\mathcal{L}_p^n(E_m) \to \mathcal{L}_p^n(E_m))$$

is called a generalized convolution if at each point $x \in \tilde{E}_m$ it is locally equivalent to some operator from w_p^n. We shall denote the space of such operators by S_p^n. For a point at infinity η, by Lemma 2, there exists a unique operator A_η locally equivalent to A at η. Moreover, the dependence of A on η is continuous in norm. At finite points, however, the representative $A_x (A \overset{x}{\sim} A_x)$ is not uniquely determined. But the value $A_x(\infty)$, by Lemma 2, is determined by the point x; moreover $A_x(\infty)$ is continuous on E_m, admits extension by continuity from E_m to $\tilde{E}_m$, and there coincides with $A_\eta(\infty)$.

4°. **Symbol of a generalized convolution.** Let $A \in S_p^n$. In the direct product $\tilde{E}_m \times \dot{E}_m$, we single out two subspaces: $\mathfrak{N} \times \dot{E}_m$ and $\tilde{E}_m \times \infty$. We denote their union by Δ. Define on Δ the matrix-function $\hat{A}(x,\xi)$ by the rule

$$\hat{A}(x,\xi) = \begin{cases} \hat{A}_x(\xi), & (x,\xi) \in \mathfrak{N} \times \dot{E}_m, \\ \hat{A}_x(\infty), & (x,\xi) \in \tilde{E}_m \times \infty. \end{cases}$$

We shall call this matrix the symbol of the operator A. From 3° it follows that the symbol of an operator $A \in S_p^n$ satisfies two conditions:

1) $\hat{A}_\eta(\xi) = \hat{A}(\eta,\xi)$ $(\eta \in \mathfrak{N},\ \xi \in \dot{E}_m)$ depends continuously on η in the metric m_p^n

$$(\|\hat{A}_\eta(\xi) - \hat{A}_{\eta_0}(\xi)\|_{m_p^n} \to 0, \quad \eta \to \eta_0).$$

In the case $p = 2$ this means continuity in the sense of uniform convergence.

2) $\hat{A}(x,\infty)$ is continuous on $\tilde{E}_m$.

From the theorem on the enveloping operator ([1,2]) one can also conclude the converse: if a matrix-function $\Phi(x,\xi)$, defined on Δ, satisfies conditions 1), 2), then there exists an operator $A (\in S_p^n)$ such that

$$\hat{A}(x,\xi) = \Phi(x,\xi).$$

Moreover, the operator A is determined up to a completely continuous summand, and the two-sided estimates hold

$$\|\hat{A}\|_p \le \|\,|A|\,\|_p \le (m+1)\|\hat{A}\|_p.$$

Here

$$\|\hat{A}\|_p = \max\left[\max_{\eta\in\mathfrak{N}} \|\hat{A}_\eta(\xi)\|_{m_p^n},\ \max_{x\in\tilde{E}_m} \|\hat{A}(x,\infty)I\|_p\right].$$

In the case $p = 2$ this

means that there is an isomorphism between the quotient ring S_p^n by the ideal of completely continuous operators and the ring of continuous matrix-functions on Δ with the topology of uniform convergence.

Theorem 1. *The operator A is a Noether operator if and only if its symbol has a determinant that nowhere on Δ vanishes.*

5°. Composite convolution. Let the space $\dot{E}_m$ be divided into a finite number of closed conic sets Γ_i*, not intersecting at interior points, and let the trace of the boundary $\left(\bigcup(\Gamma_i \setminus \operatorname{int}\Gamma_i)\right)$ on the unit sphere consist of closed, nonintersecting smooth surfaces.

An operator B of the form

$$B = \sum_{i=1}^{N} A_i P_{\Gamma_i}, \tag{2}$$

where $A_i \in S_p^n$, will be called a composite convolution. Here P_{Γ_i} are operators defined by the rule:

$$(P_{\Gamma_i} f)(x) = 0 \quad \text{for } x \notin \Gamma_i, \qquad (P_{\Gamma_i} f)(x) = f(x) \quad \text{for } x \in \Gamma_i.$$

Just as for generalized convolution, one can introduce a symbol, defining it on the parts $\Delta_i = [(\mathfrak{R} \cap \Gamma_i) \times E_m] \cup [\Gamma_i \times \infty]$ and setting it there equal to the symbol of the operator A_i. The symbol also satisfies conditions of the type 1), 2). If, conversely, a matrix-function satisfies these conditions, then there exists a composite convolution of the form (2), unique up to a completely continuous summand, with this symbol. In the case $p = 2$, the space of piecewise-continuous matrix-functions on Δ with discontinuities of the first kind on the surfaces separating the Γ_i is isomorphic to the quotient space of composite convolutions of the form (2) by the subspace of completely continuous operators.

Theorem 2. *Let $n = 1$, $m > 1$. In order that the operator B be a Noether operator, it is necessary and sufficient that its symbol nowhere on Δ_i vanish.*

6°. The Hopf-Wiener equation in the cone Γ has the form

$$(P_\Gamma K P_\Gamma f)(x) = g(x), \qquad x \in \Gamma. \tag{3}$$

Here the unknown function f and the given g belong to $\mathcal{L}_p$, and the operator K has the form (1). Equation (3) is reduced to an equation with the composite operator $B = P_{E\setminus\Gamma} + KP_\Gamma$.

Theorem 3. *If Γ is a smooth cone, then the condition $\hat{K}(\xi) \neq 0$ $(\xi \in \dot{E}_m)$ is a necessary and sufficient condition for the Noetherian property of equation (3); moreover, its index is equal to zero.*

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* By a conic set in E_m we shall mean a set which, together with each point, contains the whole ray connecting this point with the origin (including the corresponding infinitely distant point of this ray, but, possibly, excluding the origin).

Note: Figure translations are in progress. See original paper for figures.