ON THE DECIDABILITY OR UNDECIDABILITY OF CERTAIN FORMAL-LOGICAL CALCULI

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Abstract

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ON THE DECIDABILITY OR UNDECIDABILITY OF CERTAIN FORMAL-LOGICAL CALCULI

(Presented by Academician P. S. Novikov on 11 IV 1966)

In 1936 Church proved the undecidability of the predicate calculus ([1]). P. S. Novikov posed the question of the decidability or undecidability of formal-logical calculi obtained from the predicate calculus by deleting various sets of axioms and rules of inference.

Let $S(\supset, \forall)$ denote the calculus with postulates for the symbols $\supset$ and $\forall$:

1) $A \supset (B \supset A)$;
2) $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$;
3) $\dfrac{A,\ A \supset B}{B}$;
4) $\forall x A(x) \supset A(t)$, where t does not occur bound in $\forall x A(x)$;
5) $\dfrac{A \supset B(x)}{A \supset \forall x B(x)}$, where A does not contain x.

It is not hard to show that the calculus $S(\supset, \forall)$ is undecidable.

To each formula E of the predicate calculus we assign a formula E' in the following way: $E' \equiv \overline{E}$ for elementary formulas E,

$$(A \supset B)' \equiv A' \supset B', \quad (A \& B)' \equiv A' \supset \overline{B'}, \quad (A \vee B)' \equiv \overline{A'} \supset B', \quad (\overline{A})' \equiv \overline{A'},$$

$$(\exists x A(x))' \equiv \overline{\forall x A'(x)}, \quad (\forall x A(x))' \equiv \forall x A'(x).$$

If S' is a calculus with transformation rules and postulates for the symbols $\supset$, $-$, and $\forall$, then for the provability of E in the predicate calculus it is necessary and sufficient that the corresponding formula E' be provable in S'. Thus the calculus S' is undecidable.

Next, to each formula E of the calculus S' we assign a formula E^0 in the following way: $E^0 \equiv \overline{E}$ for elementary formulas E,

$$(A \supset B)^0 \equiv \overline{A^0} \supset B^0, \quad (\overline{A})^0 \equiv \overline{A^0}, \quad (\forall x A(x))^0 \equiv \overline{\forall x A^0(x)}.$$

If S^0 is a calculus with transformation rules for the symbols $\supset$, $-$, and $\forall$, with postulates for the symbols $\supset$, $\forall$, and with the axiom

$$(A \supset B) \supset ((A \supset \overline{B}) \supset \overline{A}),$$

then for the provability of the formula E in the calculus S' it is necessary and sufficient that the corresponding formula E^0 be provable in S^0. Consequently, the calculus S^0 is undecidable.

Let S^* be the calculus obtained from the calculus S^0 by adding 1) the symbol L to the alphabet; 2) the clause "L is a formula" to the basic clauses of the definition of a formula; 3) the axioms $\overline{L}$ and $\overline{A} \supset (A \supset L)$. To each formula E of the calculus S^* we assign a formula E^*, replacing all occurrences of the symbol L in E by the formula $\mathcal{A} \supset \mathcal{A}$, where $\mathcal{A}$ is some propositional letter. It can be shown that from the provability of a formula E in S^* there follows the provability of the corresponding formula E^* in S^0. Therefore the formulas of the calculus S^0 are provable in S^0 if and only if they are provable in S^*, and consequently the calculus S^* is undecidable.

Next, to each formula E of the calculus S^* we assign a formula E^+ in the following way: $E^+ \equiv E$ for elementary formulas E,

$$(A \supset B)^+ \equiv A^+ \supset B^+, \quad (\overline{A})^+ \equiv A^+ \supset L, \quad (\forall x A(x))^+ \equiv \forall x A^+(x).$$

If S^+ is the calculus obtained from the calculus S^* by deleting the axioms $\overline{L}$, $\overline{A} \supset (A \supset L)$, and

$$(A \supset B) \supset ((A \supset \overline{B}) \supset \overline{A}),$$

then for the provability of the formula E in S^* it is necessary and sufficient that the corresponding formula

E^+ was provable in S^+. Thus, the calculus S^+ is undecidable, and consequently the calculus $S(\supset, \forall)$, which differs from S^+ only in the rules for formation of formulas, is also undecidable.

If by $S(\supset, \exists)$ we denote the calculus with postulates for the symbols $\supset$ and $\exists$, then $S(\supset, \exists)$ is also **undecidable**. Moreover, every calculus obtained from the calculus $S(\supset, \forall)$ (or $S(\supset, \exists)$) by adding an arbitrary set from among the remaining postulates of the predicate calculus is **undecidable**.

However, if from the predicate calculus one deletes, one at a time, a single postulate for the symbol ⊃, then each of the three calculi obtained in this way is **decidable**. Moreover, if from the predicate calculus one deletes, one at a time, two postulates: one for the symbol ∀, the other for the symbol ∃, then again each of the four calculi obtained in this way turns out to be **decidable**.

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REFERENCES

1. S. Kleene, *Introduction to Metamathematics*, IL, 1957.

Note: Figure translations are in progress. See original paper for figures.