Branching of solutions of the Cauchy problem for a class of nonlinear integro-differential equations

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Abstract

We consider the equation

$$\frac{\partial u}{\partial t} = \int_0^1 k\left(x, y, u(y,t), \frac{\partial u(y,t)}{\partial t} t\right) dy \tag{1}$$

where

$$k(x, y, u_1, u_2 t) = \sum_{l+k+j\geq 1}^{\infty} k_{l,k,j}(x,y) u_1^l u_2^k t^j,$$

subject to the initial condition

$$u(x,t)|_{t=0} = 0. \tag{2}$$

A sufficient condition for the existence of a unique solution has been established, and the differential branching equation for problem (1), (2) has been constructed.

Bibliography: 4 items.

Full Text

Preamble

This section addresses the existence and uniqueness of solutions for a class of nonlinear integral equations. We consider the equation:

$$u(x,t) = \int_0^1 k(x, y, u(y,t), \dots, u^{(k)}(y,t), t)\, dy$$

subject to the boundary conditions $u(x,0) = 0$ and specific constraints on the derivatives at the boundaries. Following the methodology established by T. M. [3] and further developed in [1, 2], we analyze the operator properties in the space $C[R]$. We assume the kernel $k(x, y, \dots)$ is sufficiently smooth and satisfies the Lipschitz conditions:

$$|u_1| < \rho_1, \quad |u_2| < \rho_2, \quad |t| < \rho$$

where ρ defines the domain of interest.

§ 1. Existence and Uniqueness of Solutions

Let the kernel be represented by the expansion $k(x, y, u, t) = \sum k_{i,k,j}(x, y) u^i (u')^k t^j$. By applying the resolvent method to the linearized part of the equation, we transform the original problem into an equivalent integral form:

$$u(x,t) = \int_0^1 \left[B_{0,0,j}(x,y) t^j + B_{1,k,0}(x,y) u^{(k)}(y,t) + \dots \right] dy \tag{7}$$

where the coefficients $B_{i,k,j}(x, y)$ are determined by the kernel $k(x, y)$ and the associated Fredholm resolvent $\Gamma(x, y)$.

To find the solution $u(x, t)$, we seek a power series expansion in terms of t:

$$u(x,t) = \sum_{i=1}^{\infty} u_i(x) t^i \tag{9}$$

Substituting this expansion into the integral equation and equating coefficients of like powers of t, we obtain a recursive system for $u_i(x)$:

$$u_i(x) = f_i(x, u_1, \dots, u_{i-1}) \tag{12}$$

For $i = 1$, the first term is given by $u_1(x) = \int_0^1 B_{0,0,1}(x, y)\, dy$. For higher orders $i \geq 2$, the functions f_i depend on the previously determined components.

We establish the convergence of this series by the method of majorants. Let A be a constant such that $|B_{i,k,j}(x, y)| \leq A$ for all $0 \leq x, y \leq 1$. We define a sequence of constants a_i such that $|u_i(x)| \leq a_i$. By constructing a majorizing scalar equation, we show that the series $\sum a_i t^i$ converges for $|t| < \rho$, which implies the uniform convergence of the series (9) in the space $C[R]$.

Furthermore, we prove the uniqueness of this solution. Suppose there exists another solution $\omega(x, t)$ satisfying the same conditions. By considering the difference $v(x, t) = u(x, t) - \omega(x, t)$ and applying the derived estimates, we show that $v(x, t)$ must vanish identically. Specifically, as $t \to 0$, the norm of the difference is bounded by a term that approaches zero faster than any power of t, leading to $\omega(x, t) = 0$ in the limit. Thus, the problem defined by (1) and (4) has a unique solution in $C[R]$.

§ 2. Extension to Parameter-Dependent Kernels

In this section, we generalize the results to cases where the kernel depends on an additional parameter or functional form. We consider the modified equation:

$$\frac{\partial u}{\partial t} + \int_0^1 E(x,y)\frac{\partial u}{\partial y}dy = \xi(t)\phi(x) + \int_0^1 \mathcal{K}(x,y,u,u',t)\,dy \tag{26}$$

where $\xi(t)$ is a given function and $\phi(x)$ represents the spatial distribution. Using the transformation $B_{i,k,j}(x,y) = k_{i,k,j}(x,y) + \int \Gamma(x,y_1)k_{i,k,j}(y_1,y)dy_1$, we reduce the problem to a form similar to § 1.

We seek a solution in the form of a generalized series:

$$u(x,t) = \sum_{i=1}^{\infty} u_i(x,\xi,\xi',\dots,\xi^{(i-1)})t^i \tag{31}$$

The coefficients u_i are determined by solving the system of equations (32) and (33). For the case $k_0 = 1$, the first-order term is:

$$u_1(x,\xi) = \xi(t)\phi(x) + \int_0^1 B_{0,0,1}(x,y)dy$$

For $n \geq 2$, the terms u_n are calculated recursively. For example, the second-order term u_2 involves the derivatives of the parameter ξ and the nonlinear interactions of u_1:

$$u_2(x,\xi,\xi') = \frac{1}{2}\left[\int_0^1 (B_{1,0,0}u_1 + B_{0,1,1}u_1' + \dots)dy\right] \tag{41}$$

The convergence of the series (31) is guaranteed under the condition that the majorant series for the coefficients a_i converges. This allows us to state Theorem 2: if the kernel $k(x,y)$ satisfies the smoothness conditions and the parameter $\xi(t)$ is sufficiently regular, the problem (1), (4) possesses a unique solution represented by the series (31).

As a practical example, consider the case where $E(x,y) = x/y$ and $\phi(x) = x$. Substituting these into the recursive formulas, we obtain the specific components $u_1, u_2, \dots$, which demonstrate the application of the method to linear and nonlinear boundary value problems. The results confirm that the solution depends continuously on the initial parameters and the boundary conditions.

Note: Figure translations are in progress. See original paper for figures.

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