INTEGRAL REPRESENTATIONS FOR FUNCTIONS HOLOMORPHIC IN CONVEX POLYCIRCULAR DOMAINS, AND TAYLOR’ S FORMULA

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Abstract

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INTEGRAL REPRESENTATIONS FOR FUNCTIONS HOLOMORPHIC IN CONVEX POLYCIRCULAR DOMAINS, AND TAYLOR’S FORMULA

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In the present note, in the case of bounded convex complete n-circular $(n \geq 2)$ domains, a general integral representation is given (§ 1, Theorem 2), which reflects the specific properties of this class of domains and an important feature of which is its close connection with the Cauchy formula of one complex variable. Here also (§ 2), in the case of n $(n \geq 1)$ complex variables, a Taylor formula with a remainder term containing no unknown numbers is given.

§ 1. Let $f = f(z_1, \dots, z_n)$ be a function holomorphic in a complete n-circular $(n \geq 2)$ domain Q with center at the point $(0, \dots, 0)$, let k be a natural number, and let $\gamma_1, \dots, \gamma_k$ be arbitrary positive numbers satisfying $\gamma_j \geq 1$ $(j = 1, \dots, k)$. Suppose further that for each j from the set $\{1, \dots, k\}$

$$\lambda_1^{(j)}, \lambda_2^{(j)}, \dots, \lambda_{p_j}^{(j)} \qquad (1 \leq p_j \leq n)$$

are arbitrary, but all distinct, natural numbers taken from $\{1, \dots, n\}$.

Introduce the notation:

$$L_{Y_j}[f] = L_{\left(\gamma_j; \lambda_1^{(j)}, \dots, \lambda_{p_j}^{(j)}\right)}[f] = \gamma_j f + \sum_{q=1}^{p_j} z_{\lambda_q^{(j)}} f'_{z_{\lambda_q^{(j)}}}, \qquad j = 1, \dots, k,$$

$$L_Y^{(k)}[f] = L^{(k)}_{(Y_1, \dots, Y_k)}[f] = L_{Y_k}\left[L_{Y_{k-1}} \cdots [L_{Y_1}[f]] \cdots\right]*$$

and put $L_Y^{(0)}[f] = f$.

Suppose that $\varepsilon_1, \dots, \varepsilon_k$ are real numbers satisfying $0 \leq \varepsilon_j \leq 1$ $(j = 1, \dots, k)$, and that the point $z = (z_1, \dots, z_n) \in Q$. Replace in z the coordinates

$$z_{\lambda_1^{(1)}}, \dots, z_{\lambda_{p_1}^{(1)}}$$

respectively by

$$\varepsilon_1 z_{\lambda_1^{(1)}}, \dots, \varepsilon_1 z_{\lambda_{p_1}^{(1)}}.$$

Denote the result by

$$z_{\left(\lambda_1^{(1)},\dots,\lambda_{p_1}^{(1)}\right)}(\varepsilon_1)$$

(in short, $z_{M_1}(\varepsilon_1)$**). Replace in $z_{M_1}(\varepsilon_1)$

$$z_{\lambda_1^{(2)}}, \dots, z_{\lambda_{p_2}^{(2)}}$$

respectively by

$$\varepsilon_2 z_{\lambda_1^{(2)}}, \dots, \varepsilon_2 z_{\lambda_{p_2}^{(2)}}.$$

Denote the result by

$$z_{(M_1,M_2)}\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\end{pmatrix}$$

and so on. Finally, replace in

$$z_{(M_1,\dots,M_{k-1})}\begin{pmatrix}\varepsilon_1\\ \cdots\\ \varepsilon_{k-1}\end{pmatrix}$$

the coordinates

$$z_{\lambda_1^{(k)}}, \dots, z_{\lambda_{p_k}^{(k)}}$$

respectively by

$$\varepsilon_k z_{\lambda_1^{(k)}}, \dots, \varepsilon_k z_{\lambda_{p_k}^{(k)}}.$$

Denote the result by

$$z_{(M_1,\dots,M_k)}\begin{pmatrix}\varepsilon_1\\ \cdots\\ \varepsilon_k\end{pmatrix}$$

(or briefly

$$z_M\begin{pmatrix}\varepsilon_1\\ \cdots\\ \varepsilon_k\end{pmatrix}, \qquad \text{where } M=(M_1,\dots,M_k).$$

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* As here and everywhere below, $Y_j=(\gamma_j;\lambda_1^{(j)},\dots,\lambda_{p_j}^{(j)})$, $j=1,\dots,k$, $Y=(Y_1,\dots,Y_k)$.

** Similarly, here and below, $M_1=(\lambda_1^{(1)},\dots,\lambda_{p_1}^{(1)})$, $M_j=(\lambda_1^{(j)},\dots,\lambda_{p_j}^{(j)})$, $j=1,\dots,k$.

Theorem 1. If the function $f(z)$ $(n\geqslant 2)$ is holomorphic in the domain Q, then for every natural k the following formula holds in the domain Q:

$$f(z)=\int_0^1 d\varepsilon_1\dots\int_0^1 d\varepsilon_{k-1}\int_0^1 \varepsilon_1^{\gamma_1-1}\dots\varepsilon_k^{\gamma_k-1}L_Y^{(k)}\left[f\left(z_M\begin{pmatrix}\varepsilon_1\\ \cdots\\ \varepsilon_k\end{pmatrix}\right)\right]d\varepsilon_k. \qquad (1)$$

The proof is similar to the proof of Theorem 1 in the author's note (^{1}). Introduce the notation

$$L_Y^{(-k)}[f]=\int_0^1 d\varepsilon_1\dots\int_0^1 d\varepsilon_{k-1}\int_0^1 \varepsilon_1^{\gamma_1-1}\dots\varepsilon_k^{\gamma_k-1}f\left(z_M\begin{pmatrix}\varepsilon_1\\ \cdots\\ \varepsilon_k\end{pmatrix}\right)d\varepsilon_k$$

(as before, f is holomorphic in Q). By means of formula (1) it is established that

$$L_Y^{(-k)}[L_Y^{(k)}[f]]=L_Y^{(k)}[L_Y^{(-k)}[f]]=f.$$

Consequently, $L_Y^{(-k)}[f]$ is the inverse operator with respect to the operator $L_Y^{(k)}[f]$ (or, briefly, the inverse operator*).

Theorem 2.** Let $D\in(T)$, let the function $f(z)$ $(n\geqslant 2)$ be holomorphic in D, and let α be a number equal to 0 or 1. Then, if the functions $f_\nu^{(\alpha)}(z)$,

$\nu = 1, \dots, n$, and all their partial derivatives up to order μ ($\mu \geqslant 0$), inclusive, are continuous in $D \cup S$, then for $k = 0, 1, \dots, \mu$ and $z \in D$

$$f(z) = \alpha f(0) + \frac{1}{n + \alpha(1-n)} \times$$

$$\times \sum_{\nu=1}^{n} \frac{z_\nu^\alpha}{(2\pi)^n i} \int d\omega_\tau \int d\omega_\theta \int_{|\xi|=1} L_{\alpha+1, n-1}^{(n-1-\alpha)} \left[L_Y^{(-k)} \left[\frac{1}{\xi - u} \right] \right] L_Y^{(k)} \left[F_{0\nu}^{(\alpha)}(\xi, r, \theta) \right] d\xi. \tag{2}$$

In the course of the proof, for a function $f(z)$ satisfying the hypotheses of Theorem 2, the relation

$$\frac{1}{(2\pi)^n i} \int d\omega_\theta \int_{|\xi|=1} L_{\alpha+1, n-1}^{(n-1-\alpha)} \left[\frac{1}{\xi - u} \right] F_{0\nu}^{(\alpha)}(\xi, r, \theta)\, d\xi =$$

$$= \frac{1}{(2\pi)^n i} \int d\omega_\theta \int_{|\xi|=1} L_{\alpha+1, n-1}^{(n-1-\alpha)} \left[L_Y^{(-k)} \left[\frac{1}{\xi - u} \right] \right] L_Y^{(k)} \left[F_{0\nu}^{(\alpha)}(\xi, r, \theta) \right] d\xi$$

($k = 0, 1, \dots, \mu$ and $z \in D$) is established. Then formula (4.15) from the author' s (2) is used.

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* In the author' s papers (1,2), the suitably defined expressions

$$K_{\binom{\gamma_1}{\gamma_k}}(\xi, z), \quad P_{\binom{\gamma_1}{\gamma_k}}(\rho, \varphi - \psi), \quad S_{\binom{\gamma_1}{\gamma_k}}(e^{i\varphi}, z),$$

as is established by means of formulas (1.18), (1.20) from (2) (the first of these formulas also occurs in paper (1) as formula (1)), are the inverse operators with respect, respectively, to the operators

$$L_{\binom{\gamma_1}{\gamma_k}}^{(k)} \left[\frac{1}{\xi - z} \right], \quad J_{\binom{\gamma_1}{\gamma_k}}^{(k)} \left[\frac{1 - \rho^2}{1 + \rho^2 - 2\rho\cos(\varphi - \psi)} \right], \quad L_{\binom{\gamma_1}{\gamma_k}}^{(k)} \left[\frac{e^{i\varphi} + z}{e^{i\varphi} - z} \right];$$

therefore it is natural to denote these inverse operators respectively by

$$L_{\binom{\gamma_1}{\gamma_k}}^{(-k)} \left[\frac{1}{\xi - z} \right], \quad J_{\binom{\gamma_1}{\gamma_k}}^{(-k)} \left[\frac{1 - \rho^2}{1 + \rho^2 - 2\rho\cos(\varphi - \psi)} \right], \quad L_{\binom{\gamma_1}{\gamma_k}}^{(-k)} \left[\frac{e^{i\varphi} + z}{e^{i\varphi} - z} \right].$$

Analogously also in the case of the expressions $K_{l,l+k-1}(\xi, z)$, $P_{l,l+k-1}(\rho, \varphi - \psi)$, $S_{l,l+k-1}(e^{i\varphi}, z)$. The same applies when z is replaced by u.

** In formulating this theorem we adhere to the notation used in (1–4).

We point out one feature of the integral representation (2). From formula (2) it is seen that the integral

$$\frac{1}{2\pi i}\int_{|\zeta|=1} L^{(n-1-\alpha)}_{\alpha+1,n-1}\left[L^{(-k)}_{Y}\left[\frac{1}{\zeta-u}\right]\right] L^{(k)}_{Y}\left[F^{(\alpha)}_{0\nu}(\zeta,r,\theta)\right]\,d\zeta$$

is

$$L^{(n-1-\alpha)}_{\alpha+1,n-1}\left[L^{(-k)}_{Y}\left[\frac{1}{2\pi i}\int_{|\zeta|=1}\frac{1}{\zeta-u}L^{(k)}_{Y}\left[F^{(\alpha)}_{0\nu}(\zeta,r,\theta)\right]\,d\zeta\right]\right],$$

i.e., the given integro-differential operator of the Cauchy integral, regarded as a function of the variables $z_1,\dots,z_n$.

Remark 1. Along with formula (2) there also hold two further general formulas, one of which has a feature analogous to that of formula (2), but with respect to the Poisson integral, and the other with respect to the Schwarz integral.

Remark 2. Leaving aside the above-indicated feature of the integral representation (2), we note that (similarly to Theorem 5 (the case of the domains B and H in (1))) in the case of a bounded complete n-circular ($n \geq 2$) domain (under insignificant restrictions on its boundary), one can obtain integral formulas similar in character to formula (2). In doing so, alongside the corresponding integral representations, one uses formula (1) of the present note and formula (2) from (3).

Remark 3. Formula (1) (for $k=1$) remains valid also in the case when γ_1 is any positive number. But for $0<\gamma_1<1$ the integral entering this formula should be understood as improper. Taking account of the same remark concerning the analogous integrals, Theorem 1 and the entire contents of § 1 connected with $\gamma_1,\dots,\gamma_k$ remain valid also in the case when $\gamma_1,\dots,\gamma_k$ are arbitrary positive numbers.

§ 2. Let G be a star-shaped domain with respect to the point $(z_1^0,\dots,z_n^0)$* in the space C^n of complex variables $z_1,\dots,z_n$, $n\geq 1$, and let $f=f(z_1,\dots,z_n)$ be a function holomorphic in G. In what follows the notations

$$\frac{\partial^k_\nu f}{\partial^k}=\frac{\partial^k f}{\partial z_\nu^k}\quad(\nu=1,\dots,n),\qquad \frac{\partial^k f}{\partial_1^{\alpha_1}\dots\partial_n^{\alpha_n}}=\frac{\partial^k f}{\partial z_1^{\alpha_1}\dots\partial z_n^{\alpha_n}}\quad(\alpha_1+\dots+\alpha_n=k)$$

will be convenient.

Theorem 3. If the function $f(z_1,\dots,z_n)$ ($n\geq 1$) is holomorphic in the domain G, then for every natural m in G the formula** holds

$$f(z_1,\dots,z_n)=\sum_{k=0}^{m}\left(\sum a^{(k)}_{\alpha_1\dots\alpha_n}(z_1-z_1^0)^{\alpha_1}\dots(z_n-z_n^0)^{\alpha_n}\right)+R_m(z_1,\dots,z_n),$$

where

$$a^{(k)}_{\alpha_1\dots\alpha_n}=\frac{1}{\alpha_1!\dots\alpha_n!}f^{(k)}_{z_1^{\alpha_1}\dots z_n^{\alpha_n}}(z_1^0,\dots,z_n^0),***$$

$$R_m(z_1,\dots,z_n)=\frac{1}{(m-1)!}\int_0^1 d\varepsilon\int_0^{\varepsilon}(\varepsilon-t)^{m-1}\left(\sum_{\nu=1}^{n}(z_\nu-z_\nu^0)\frac{\partial}{\partial_\nu}\right)^{m+1}f(Z_1,\dots,Z_n)\,dt$$

$$(Z_1=z_1^0+t(z_1-z_1^0),\dots,Z_n=z_n^0+t(z_n-z_n^0),\ \varepsilon\text{ and }t\text{ are real}).$$

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* A domain is called star-shaped with respect to the point $(z_1^0,\dots,z_n^0)$ if, together with each point, it contains the whole segment joining this point to the point $(z_1^0,\dots,z_n^0)$.

** In the inner sum the summation extends over all groups of nonnegative integers $\alpha_1,\dots,\alpha_n$ satisfying the condition $\alpha_1+\dots+\alpha_n=k$.

*** We take $0!=1$.

In the proof, the formula (2) from (3) obtained by the author is used essentially (this formula is taken here in the case when G is a star-shaped domain with respect to the point $(z_1^0,\dots,z_n^0)$), as is the analogous formula.

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