ON THE REFLEXIVITY OF SOME COMBINATIONS OF PARTIALLY ORDERED SPACES

MATHEMATICS

1967

SovietRxiv

Abstract

Full Text

UDC 513.88 : 517.4./5

MATHEMATICS

I. P. KOSTENKO

ON THE REFLEXIVITY OF SOME COMBINATIONS OF PARTIALLY ORDERED SPACES

(Presented by Academician L. V. Kantorovich on 25 I 1967)

1°. In the theory of K-spaces, an essential role is played by an operation, introduced by A. G. Pinsker, which makes it possible to construct, from a given family of K-spaces X_ξ ($\xi \in \Xi$), numerous new K-spaces. This is the operation of **combination**, generalizing the direct-sum operation known in algebra. It is multivalued. The combination uniquely defined in ([1]) we call **complete**, in accordance with the terminology of ([2])*. It is the largest of all possible combinations; the others are normal subspaces of the complete one containing all X_ξ ($\xi \in \Xi$). Notation: $S_{\xi\in\Xi}X_\xi$.

The smallest of the possible combinations we call **finite**–this is the direct sum of the spaces with coordinatewise order. In other words, the combination $X = SX_\xi$ is called **finite** if, for any $x \in X$, $x = Sx_\xi$ ($x_\xi \in X_\xi$), $x_\xi \neq 0$ only for a finite number of indices ξ.

In this note the following problem is posed: given a family of reflexive (with respect to order) K-spaces X_ξ ($\xi \in \Xi$); which combinations of these spaces are reflexive, and which are not? A particular case of combinations is that of discrete K-spaces, considered as combinations of one-dimensional spaces; therefore the question posed also contains another: what are the conditions for reflexivity of a discrete K-space? This question intersects with an analogous question from the theory of Köthe sequence spaces: which spaces are perfect (see ([3]))? The concept of a reflexive K-space is close to the concept of a Köthe space introduced in ([4]).

Reflexivity (with respect to order) of a K-space X was first considered by H. Nakano (see ([5])). He showed that if in X there exists a sufficient set of fully linear functionals, then under the canonical embedding of X into $\overline{X}^{**}$ by means of the correspondence $x \to F_x$, $F_x(f) = f(x)$, $x \in X$, $f \in \overline{X}$, X turns out to be a normal subspace in $\overline{X}$. If all of $\overline{X}$ is exhausted by functionals of the form F_x, then the K-space X is called **reflexive*****. H. Nakano also established a certain general criterion of reflexivity (see ([1]), p. 290).

Examples of reflexive combinations are the K-spaces l^p $(p \geq 1)$, m, and $l[\Xi]$—the space of functions nonzero at no more than a countable set of points of Ξ and such that

$$\sum_{\xi \in \Xi} |x(\xi)| < +\infty.$$

Examples of nonreflexive combinations are the space c_0 of sequences converging to zero, and $m[\Xi]$ (Ξ uncountable)—the K-space—

---

* The elements of the complete combination are all possible families $\{x_\xi\}_{\xi \in \Xi}$, $x_\xi \in X_\xi$. For example, the space of all real sequences s is the complete combination of one-dimensional spaces. In what follows we use, without special explanation, the terminology from ([1]).

** $\overline{X}$ is the K-space of fully linear functionals on X.

*** Reflexivity with respect to order differs from reflexivity with respect to the norm. Thus, for the KN-space m, the order dual is l, and it does not coincide with the norm dual.

of functions bounded with respect to Ξ and different from zero at no more than a countable set of points.

2°. It is quite easy to show that complete and finite sums of reflexive spaces are reflexive. With the aid of Theorems IX.4.6 and IX.4.7 from [1] one can prove that the KN-space of bounded elements with a sufficient set of completely linear functionals is reflexive. It is interesting that such a space is not norm-reflexive, except in the finite-dimensional case.

Definition 1. Let $X = S_{\xi \in \Xi} X_\xi$ and $\Xi' \subset \Xi$. The set X' of all elements $x \in X$ of the form $x = Sx_\xi$, where $x_\xi = 0$ for $\xi \in \overline{\Xi'}$, forms a component in X. We shall call it a **subsum** of the sum X. Notation: $X' = S_{\xi \in \Xi'} X_\xi$.

Definition 2. We shall say that the sum $X = S_{\xi \in \Xi} X_\xi$ has **property** (R_0) if in it one cannot choose a system of elements $\{y_\xi\}_{\xi \in \Xi}$ such that: 1) $y_\xi \in X_\xi^+$; 2) the system is unbounded, and 3) in every reflexive subsum $X' = S_{\xi \in \Xi'} X_\xi$ there exists $y' = S_{\xi \in \Xi'} y_\xi$.

The space $l[0,1]$ has this property, while $m[0,1]$ does not.

Theorem 1. *In order that the sum $X = SX_\xi$, where all X_ξ are reflexive, be reflexive, it is necessary and sufficient that it have property (R_0).*

This theorem helps to distinguish two types of sums for which the question of reflexivity is solved more simply.

Definition 3. Let $X = S_{\xi\in\Xi} X_\xi$, and let $\{\Xi_\beta\}_{\beta\in B}$ be the collection of all countable subsets of Ξ. If every subsum $X^{(\beta)} = S_{\xi\in\Xi_\beta} X_\xi$ $(\beta \in B)$ is reflexive, we shall call X a **countably reflexive** sum.

Definition 4. We shall say that the sum $X = S_{\xi\in\Xi} X_\xi$ has **property** (R_1) if in it one cannot choose a system $\{y_\xi\}_{\xi\in\Xi}$ such that: 1) $y_\xi \in X_\xi^+$; 2) the system is unbounded, and 3) in every subsum $X' = S_{\xi\in\Xi'} X_\xi$, where Ξ' is countable, there is an element $y' = S_{\xi\in\Xi'} y_\xi$.

Let us note that if X is a countably reflexive sum, then formally property (R_1) is stronger for it than property (R_0). The space $l[0,1]$ has property (R_1), while $m[0,1]$ does not.

Theorem 2. *In order that a countably reflexive sum $X = SX_\xi$ be reflexive, it is necessary and sufficient that it have property (R_1).*

Thus, if $X = |SX_\xi|$ is not reflexive, then either it is countably reflexive, and then Theorem 2 indicates the reason for the failure of its reflexivity, or it is not countably reflexive, and then some subsum $X' = S_{\xi\in\Xi'} X_\xi$, Ξ' countable, is not reflexive. Hence one may restrict oneself to the investigation of sums of a countable set of reflexive spaces. We shall formulate this theorem in other terms.

Theorem 2′. *In order that a countably reflexive sum $X = S_{\xi\in\Xi} X_\xi$ be reflexive, it is necessary and sufficient that it contain no linear substructure, isomorphic (linearly and structurally) to the space $m[\Xi']$ (Ξ' uncountable) and situated in X so that, if y_ξ $(\xi \in \Xi')$ are the elements of X corresponding under this isomorphism to the unit elements $e_\xi \in m[\Xi']$, then: 1) $y_\xi \in X_\xi^+$ and 2) $\{y_\xi\}_{\xi\in\Xi'}$ is unbounded.*

In the necessity part, the proofs of both theorems go through for arbitrary sums.

3°. **Definition 5.** Let Y be a K-space, and let $\{X_\alpha\}_{\alpha\in A}$ be some system of its normal subspaces. The collection X of all such

elements $x \in Y$ which are representable in the form

$$x = \sum_{i=1}^{k} x_{\alpha_i}, \quad x_{\alpha_i} \in X_{\alpha_i} *$$

we shall call the **sum** of the spaces X_α $(\alpha \in A)$. Notation:

$$X = \sum_{\alpha\in A} X_\alpha.$$

The sum is also a K-space.

Definition 6. Let

$$X = \underset{\xi \in \Xi}{S} X_\xi.$$

If there exists a system of subsets $\{\Xi_\beta\}_{\beta \in B}$ of the set Ξ such that each subconnection

$$X^{(\beta)} = \underset{\xi \in \Xi_\beta}{S} X_\xi$$

is reflexive and

$$X = \sum_{\beta \in B} X^{(\beta)},$$

we shall call X a **quasi-reflexive connection** of the spaces X_ξ ($\xi \in \Xi$), and the system $\{\Xi_\beta\}_{\beta \in B}$ the **carrier** of this connection.

Let us note that a single quasi-reflexive connection may have many different carriers, but we assume that some carrier has been chosen and fixed. Any subconnection

$$X' = \underset{\xi \in \Xi'}{S} X_\xi$$

of the quasi-reflexive connection

$$X = \underset{\xi \in \Xi}{S} X_\xi$$

with carrier $\{\Xi_\beta\}_{\beta \in B}$ may be considered as an independent quasi-reflexive connection with carrier $\{\Xi_\beta \cap \Xi'\}_{\beta \in B}$. The question of the reflexivity of these connections is completely settled by Theorems 3, 4, and 5.

Definition 7. We shall call the carrier $\{\Xi_\beta\}_{\beta \in B}$ of the quasi-reflexive connection

$$X = \underset{\xi \in \Xi}{S} X_\xi$$

saturated if: 1) among the Ξ_β ($\beta \in B$) there is no finite number covering Ξ, and 2) whatever infinite $\Xi' \subset \Xi$ is taken, there exists Ξ_β such that $\Xi' \cap \Xi_\beta$ is infinite.

Theorem 3. *If, in the connection*

$$X = \underset{\xi \in \Xi}{S} X_\xi$$

with carrier $\{\Xi_\beta\}_{\beta\in B}$, *every subconnection*

$$X' = \underset{\xi\in\Xi'}{S} X_\xi,$$

whose carrier is saturated, is reflexive, then X *itself is reflexive.*

Corollary. *For the reflexivity of the connection*

$$X = \underset{\xi\in\Xi}{S} X_\xi$$

with carrier $\{\Xi_\beta\}_{\beta\in B}$, *it is sufficient that it contain no subconnection*

$$X' = \underset{\xi\in\Xi'}{S} X_\xi,$$

whose carrier $\{\Xi_\beta \cap \Xi'\}_{\beta\in B}$ *is saturated.*

Let us note that the condition of the corollary concerns only the carrier. A trivial example of a carrier possessing the indicated property is furnished by the system of all finite subsets of an infinite set.

Definition 8. Let

$$X = \underset{\xi\in\Xi}{S} X_\xi$$

be a quasi-reflexive connection with carrier $\{\Xi_\beta\}_{\beta\in B}$. We shall say that it has **property** (R_2), if in it one cannot choose a system $\{y\}_{\xi\in\Xi}$ such that: 1) $y_\xi \in X_\xi^+$; 2) the system is unbounded, and 3) in each subconnection

$$X^{(\beta)} = \underset{\xi\in\Xi_\beta}{S} X_\xi$$

there is an element

$$y^{(\beta)} = \underset{\xi\in\Xi_\beta}{S} y_\xi.$$

Theorem 4. *In order that a quasi-reflexive connection*

$$X = \underset{n\in N}{S} X_n$$

(N *is the natural series), whose carrier* $\{N_\beta\}_{\beta\in B}$ *is saturated, be reflexive, it is necessary and sufficient that it have property* (R_2).

Examples. Consider all possible systems of almost disjoint*** subsets of N. Introduce in this collection the ordering by inclusion. Zorn' s lemma guarantees the existence of maximal systems–

---

* This representation is unique for each $x \in X$ only in the case when the X_α ($\alpha \in A$) are pairwise disjoint. In this case the sum becomes a finite connection.

** A finite connection

$$X = \underset{n \in N}{S} X_n$$

with carrier $\{n\}_{n \in N}$ does not have this property, whereas with carrier $\{N\}$ it does.

*** N' is almost disjoint from N'' if $N' \cap N''$ is finite or empty.

system, while among them there are saturated ones. Let $T = \{N_\beta\}_{\beta \in B}$ be such. Consider m as a union of one-dimensional X_n ($n \in N$) and single out the system of subunions

$$m^{(\beta)} = \sum_{n \in N_\beta} X_n \quad (\beta \in B).$$

The quasireflexive union

$$m[\dot{T}] = \sum_{\beta \in B} m^{(\beta)}$$

does not possess property (R_2), and therefore is not reflexive.

Similarly one constructs $s[\dot{T}]$, $l[\dot{T}]$, etc. An example of a carrier not satisfying the condition in the corollary to Theorem 3 is provided by the system $\{N_x\}_{x \in (0,1)}$ of almost disjoint sets, defined by Sierpiński in ([6]), supplemented by the set

$$N_0 = N \setminus \bigcup_{x \in (0,1)} N_x.$$

Denote it by $T^{(S)}$ and, as above, construct the reflexive spaces $s[T^{(S)}]$, $m[T^{(S)}]$, and $l[T^{(S)}]$. These spaces will also be perfect sequence spaces of Köthe. The space δ, introduced in ([7]), is a quasireflexive union with a saturated carrier, not possessing property (R_2) (see also ([3])).

The presence of property (R_2) in the quasireflexive union $X = SX_n$ can be connected with the space $m[\dot{T}]$.

Theorem 4. *In order that the quasireflexive union*

$$X = \sum_{n \in N} X_n$$

be reflexive, it is necessary and sufficient that it contain no linear substructure isomorphic to the space $m[\dot{T}]$ *and situated in* X *in such a way that, if* y_{n_k} $(k = 1, 2, ...)$ *are the elements of* X *corresponding under this isomorphism to the vectors* $e_k \in m[\dot{T}]$*, then: 1)* $y_{n_k} \in X_{n_k}+$ $(n_k$ *are distinct for* $k = 1, 2, ...)$*; 2) the system* $\{\lambda_k y_{n_k}\}_{k=1,2,...}$ *is unbounded for arbitrary numbers* $\lambda_k \neq 0$*.*

This theorem also contains one more necessary criterion for the reflexivity of an arbitrary union, since in proving the necessity of the condition of the theorem the quasireflexivity of X is not used. From Theorems 2, 3, and 4 it follows easily that

Theorem 5. *In order that the quasireflexive union*

$$X = \sum_{\xi \in \Xi} X_\xi$$

with carrier $\{\Xi_\beta\}_{\beta \in B}$ *be reflexive, it is necessary and sufficient that: 1)* X *possess property* (R_1)*, and 2) every subunion*

$$X' = \sum_{\xi \in \Xi'} X_\xi,$$

for which Ξ' *is countable and the carrier is saturated, possess property* (R_2)*.*

4°. Let us formulate a third necessary criterion for the reflexivity of an arbitrary union.

Definition 9. We shall say that the union

$$X = \sum_{\xi \in \Xi} X_\xi$$

possesses **property** (R_3) if in it one cannot choose a system $\{y_n\}_{n \in N}$ such that: 1) $y_n \in X_{\xi_n}$, the ξ_n being pairwise distinct; 2) the system is unbounded, and 3) for every sequence of numbers $\lambda_n \to 0$, the element

$$\sum_{n \in N} \lambda_n y_n$$

is contained in X.

Analogously to the preceding, this property is connected with the absence in X of a substructure linearly and structurally isomorphic to the space c_0 and situated in X in a certain way.

Theorem 6. *In order that the union $X = SX_\xi$ be reflexive, it is necessary that it possess property* (R_3).

From the three criteria noted above one easily obtains three necessary criteria for the reflexivity of an arbitrary K-space.

The author expresses gratitude to his scientific adviser, Prof. B. Z. Vulikh, for his attention to the present work.

Leningrad Pedagogical Institute
named after A. I. Herzen

Received
21 I 1967

CITED LITERATURE

1. B. Z. **Vulikh**, *Introduction to the Theory of Semi-Ordered Spaces*, 1961.
2. L. V. **Kantorovich**, B. Z. **Vulikh**, A. G. **Pinsker**, *Functional Analysis in Semi-Ordered Spaces*, 1950.
3. R. **Cook**, *Infinite Matrices and Sequence Spaces*, 1960.
4. J. **Dieudonné**, J. Analyse Math., **1**, 81 (1951).
5. H. **Nakano**, *Modulared Semi-Ordered Linear Spaces*, Tokyo, 1950.
6. W. **Sierpiński**, *Cardinal and Ordinal Numbers*, Warszawa, 1958.
7. G. **Köthe**, O. **Toeplitz**, J. reine u. angew. Math., **171**, 193 (1934).

Note: Figure translations are in progress. See original paper for figures.