Some problems on the quieting of a linear system

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Abstract

The problems of choosing controls $u_1(t)$ and $u_2(t)$ that bring linear systems

$$\frac{dx}{dt} = Ax + bu_1, \quad \frac{dx}{dt} = Ax + bu_2$$

to the equilibrium state $x = 0$ in a given time $a \leq t \leq T$ are considered, subject to the minimization of the following control intensity estimates:

$$J(u_1^0) = \max\left\{\max_\tau |u_1^0(\tau)|, \nu \int_0^T |u_1^0(\tau)|\, d\tau\right\} = \min,$$

$$J(u_2^0) = \max\left\{\max_\tau |u_{2k}^0(\tau)|, \nu \int_0^T \sum_{s=1}^r |u_{2s}^0(\tau)|\, d\tau\right\} = \min,$$

$$(k = 1, \dots, r) \quad (\nu T > 1).$$

Bibliography: 9 items.

Full Text

Optimal Control with Combined Constraints

1. Problem Statement

Consider a linear dynamical system described by the following n-dimensional differential equation:

$$\dot{x} = Ax + bu(t) \tag{1.1}$$

where x is an n-dimensional state vector, $u(t)$ is a scalar control function, A is a constant $n \times n$ matrix, and b is a constant n-dimensional vector. We assume the system is controllable, satisfying the condition:

$$\operatorname{rank}[b, Ab, \dots, A^{n-1}b] = n \tag{1.2}$$

The objective is to determine an optimal control $u^0(t)$ that transfers the system from an initial state x_0 to the origin in a fixed time $T > 0$, while minimizing a combined functional of the form:

$$J(u) = \max\left\{\max_{t\in[0,T]}|u(t)|, \nu\int_0^T |u(t)|dt\right\} \to \min \tag{1.3}$$

where $\nu > 1/T$ is a weighting coefficient. This problem formulation seeks a balance between the peak magnitude of the control signal and its total L_1 consumption.

2. Properties of the Optimal Control

The optimal control $u^0(t)$ for the functional (1.3) is characterized by a specific structure. It can be shown that for the optimal solution, the two terms within the maximum operator are equal:

$$\max_t |u^0(t)| = \nu\int_0^T |u^0(t)|dt \tag{2.1}$$

If we denote $G(u) = \int_0^T |u(t)|dt$, the problem can be viewed as finding a control that minimizes the L_1 norm subject to a constraint on the maximum amplitude. Specifically, if there exists a control $u(t)$ such that $|u(t)| < \max_t |u^0(t)|$, then by the properties of L-problems in functional analysis, the integral $\int |u(t)|dt$ must increase to satisfy the boundary conditions.

The optimal control $u^0(t)$ is related to the solution of the L-problem. Let $g \subset [0, T]$ be the support of the control. The optimal control takes the form:

$$u^0(t) = \begin{cases} L \text{ sign } \sum_{j=1}^n l_j h_j(t), & t \in g \\ 0, & t \notin g \end{cases} \tag{2.16}$$

where $h_j(t)$ are the components of the fundamental matrix solution and L is the minimum possible maximum amplitude. The set g is chosen such that the condition (2.1) is satisfied.

3. Generalization to Weighted Functionals

The problem can be generalized by introducing a weighting function $\psi(t)$ into the integral constraint:

$$J(u) = \max\left\{\max_t |u|, \int_0^T \psi(t)|u(t)|dt\right\} \to \min \tag{3.1}$$

where $\psi(t)$ is a known positive function satisfying:

$$\int_0^T \psi(t)dt > 1 \tag{3.2}$$

This allows for the penalization of control efforts at different stages of the process. The optimality conditions for this case follow a similar logic to the unweighted version, requiring the equalization of the peak value and the weighted integral.

4. Systems with External Disturbances

Consider the non-homogeneous system:

$$\dot{x} = Ax + bu + q(t) \tag{4.1}$$

where $q(t)$ represents a known external disturbance. The control task remains the same: reaching the origin at time T. The influence of the disturbance can be accounted for by shifting the target state in the reachable set. The effective initial condition for the homogeneous part of the system is modified by the integral of the disturbance:

$$c_i = x_{i0} - \int_0^T \sum_{j=1}^{n} f_{ij}(T-\tau)q_j(\tau)d\tau \tag{4.2}$$

where f_{ij} are the elements of the transition matrix.

5. Multi-dimensional Control

In the case of a vector control $u = \{u_1, \dots, u_r\}$, the system is governed by:

$$\dot{x} = Ax + Bu \tag{5.1}$$

where B is an $n \times r$ matrix. The optimal control components $u_k^0(t)$ are determined by:

$$u_k^0(t) = \begin{cases} L \text{ sign } \sum_{i=1}^{n} l_i h_{ik}(t), & t \in \Delta_k \\ 0, & t \notin \Delta_k \end{cases} \tag{5.3}$$

where Δ_k are the respective supports for each control component, chosen to satisfy the combined norm constraints for the vector case.

References

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Note: Figure translations are in progress. See original paper for figures.