APPLICATION OF HAAR SERIES TO ERROR ESTIMATION IN THE COMPUTATION OF INFINITE-DIMENSIONAL INTEGRALS

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Abstract

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MATHEMATICS

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APPLICATION OF HAAR SERIES TO ERROR ESTIMATION IN THE COMPUTATION OF INFINITE-DIMENSIONAL INTEGRALS

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We estimate the error of the approximation

$$\int_{K_\infty} f(X)\,dX \approx \frac{1}{N}\sum_{\nu=1}^{N} f(X_\nu), \tag{*}$$

where K_∞ is the infinite-dimensional unit cube; $X = (x_1, x_2, \dots, x_n, \dots)$ is a point of this cube. The measure on K_∞ is defined by means of the infinite product of Lebesgue measures ($dX = dx_1 dx_2 \dots dx_n \dots$).

In (1) a sequence of points $\{X_\nu^*\}$ was constructed, called a generalized Halton sequence, whose points can be used as nodes in $(*)$, and an error estimate for such an approximation was obtained for certain classes of functions.

In § 2 of the present paper the error of the approximation $(*)$ is estimated for broader classes of functions S_p and $\bar{H}_\alpha$ (the classes H_α are analogues of the Lipschitz classes $\mathrm{Lip}\,\alpha$, $0 < \alpha \leq 1$). A new sequence of points $\{X_\nu^{**}\}$, called a generalized $_\tau$-sequence, is constructed, for which estimates are obtained that are somewhat better than for $\{X_\nu^*\}$. On the classes S_p the order of these estimates turns out to be the best possible. The concept of a grid $\{X_1, \dots, X_N\}$ with a multiplicative estimate of nonuniformity is introduced. For such grids an estimate of the approximation $(*)$ is obtained in general form.

The paper uses the method of Haar series, which made it possible to obtain error estimates on the classes S_p and H_α in the n-dimensional case ($^{2-4}$). However, in order to be able to pass to the limit as $n \to \infty$, it was necessary to improve these estimates somewhat. This is done in § 1.

In what follows, the symbol $K_{i_1 \dots i_s}$ denotes the s-dimensional face of the cube K_∞ on which $x_{i_1}, \dots, x_{i_s}$ range from 0 to 1, while all the remaining $x_i \equiv 1$.

§ 1. The case of n variables.

1.1. Expansion of a function in "different-dimensional" terms.

Consider a function $f(P)$, where $P = (x_1, \dots, x_n)$, defined in the n-dimensional unit cube $K \equiv K_{12\dots n}$. Write its expansion in the Fourier-Haar series, explicitly separating the first function of the Haar system $\chi_1(x) \equiv 1$. Let

$$C^{i_1 \dots i_s}_{k_1 \dots k_s} = \int_K f(P) \chi_{k_1}(x_{i_1}) \cdots \chi_{k_s}(x_{i_s})\, dP,$$

$$(dP = dx_1 \cdots dx_n),$$

where $1 \le i_1 < i_2 < \cdots < i_s \le n,\ 1 \le s \le n$. Then

$$f(P) = c_0 + \overset{\wedge}{\sum_{k_1,\dots,k_s}} \sum C^{i_1 \dots i_s}_{k_1 \dots k_s} \chi_{k_1}(x_{i_1}) \cdots \chi_{k_s}(x_{i_s}), \tag{1}$$

where the symbol $\hat{\sum}$ denotes summation over all different s-index quantities for $s = 1, 2, \dots, n$:

$$\hat{\sum} T_{i_1 \dots i_s} = \sum_{i_1=1}^{n} T_{i_1} + \sum_{1 \le i_1 < i_2 \le n} \sum T_{i_1 i_2} + \cdots + T_{12\dots n}.$$

In (1), as throughout what follows, the indices k_σ vary from 2 to ∞. Each of the sums following the sign $\hat{\sum}$ depends only on the s variables $x_{i_1}, \dots, x_{i_s}$.

1.2. Estimation of the integration error. Choose an integration net Σ, consisting of arbitrary points $P_1, \dots, P_N$ of the cube K, and estimate the error

$$\delta(f; \Sigma) = \frac{1}{N} \sum_{\nu=1}^{N} f(P_\nu) - \int_K f(P)\, dP. \tag{2}$$

Suppose that the series (1) converges uniformly, and substitute it into (2). After transformations entirely analogous to the transformations of Sec. 1 from (^{3}), we obtain the estimate

$$|\delta(f; \Sigma)| \le \frac{1}{N} \hat{\sum} A_p^{i_1 \dots i_s}(f)\, \Phi_q^{i_1 \dots i_s}(\Sigma), \tag{3}$$

in which $1/p + 1/q = 1$;

$$A_p^{i_1 \dots i_s}(f) = \sum_m 2^{(m_1-1)/2 + \cdots + (m_s-1)/2} \left\{ \sum_j |C^i_k|^p \right\}^{1/p},$$

and $\Phi_q^{i_1 \dots i_s}(\Sigma)$ is the s-dimensional discrepancy of the projections of the points $P_1, \dots, P_N$ onto $K_{i_1 \dots i_s}$ ([3]). Here $i = (i_1, \dots, i_s)$, $k = (k_1, \dots, k_s)$, $m = (m_1, \dots, m_s)$, $j = (j_1, \dots, j_s)$; $1 \leq m_\sigma < \infty$, $1 \leq j_\sigma \leq 2^{m_\sigma - 1}$.

In ([3]) the norm $\|f\|_p = \hat{\sum} A_p^{i_1 \dots i_s}(f)$ was used. Therefore the estimate (3) was coarsened and written in the form

$$|\delta(f;\Sigma)| \leq N^{-1} \|f\|_p \, \varphi_q(\Sigma), \quad \text{where } \varphi_q(\Sigma) = \max \Phi_q^{i_1 \dots i_s}(\Sigma).$$

Below only the simplest quantities $\Phi_\infty^{i_1 \dots i_s}(\Sigma)$ are used; we shall call them the discrepancies of the net Σ. It is easy to prove that for any net

$$\Phi_q^{i_1 \dots i_s}(\Sigma) \leq N^{1/q} \Phi_\infty^{i_1 \dots i_s}(\Sigma). \tag{4}$$

1.3. Classes of functions S_p and H_α.

Definition. A function $f(x_1, \dots, x_n)$ belongs to the class $S_p(L_{i_1 \dots i_s})$, $1 \leq p < \infty$, if for any $1 \leq i_1 < i_2 < \dots < i_s \leq n$ and $1 \leq s \leq n$

$$A_p^{i_1 \dots i_s}(f) \leq L_{i_1 \dots i_s}. \tag{5}$$

The constants $\{L_{i_1 \dots i_s}\}$ are called the defining constants of the class.

As in ([2],[3]), it is proved that if $f \in S_p$, then the series (1) converges uniformly in K, and estimate (3) with $L_{i_1 \dots i_s}$ in place of $A_p^{i_1 \dots i_s}(f)$ is an exact estimate on S_p (for any prescribed net Σ).

Definition. A function $f(x_1, \dots, x_n)$ belongs to the class $H_\alpha(L_{i_1 \dots i_s})$, $0 < \alpha \leq 1$, if for any $1 \leq i_1 < i_2 < \dots < i_s \leq n$ and $1 \leq$

$\leqslant s \leqslant n$ in the cube K

$$\left|\Delta_{\xi_{i_1}} \cdots \Delta_{\xi_{i_s}} f(P)\right| \leqslant L_{i_1 \dots i_s} \left(\frac{\alpha+1}{2}\right)^s \left|\xi_{i_1} \cdots \xi_{i_s}\right|^\alpha. \tag{6}$$

Just as in ([2]), it is proved that if $f \in H_\alpha(L_{i_1 \dots i_s})$ and $\alpha p > 1$, then

$$A_p^{i_1 \dots i_s}(f) \leqslant L_{i_1 \dots i_s} \left(2^{1+\alpha} - 2^{1+1/p}\right)^{-s}. \tag{7}$$

From (3), (4), and (7) one can obtain an estimate of $|\delta(f;\Sigma)|$ on the classes H_α, quite analogous to estimate (8) from ([3]).

§ 2. The case of infinitely many variables

2.1. Classes of functions

The classes $S_p(L_{i_1\dots i_s})$ and $H_\alpha(L_{i_1\dots i_s})$ are defined in the same way as in the n-dimensional case: the corresponding inequalities (5) or (6) must be satisfied in K_∞ for arbitrary $1 \leqslant i_1 < i_2 < \dots < i_s$, $1 \leqslant s < \infty$. In addition, it is assumed that at each point $f(X) = \lim_{n\to\infty} f(x_1, \dots, x_n, 1, 1, 1, \dots)$, and passage to the limit is possible:

$$\int_{K_\infty} f(X)\,dX = \lim_{n\to\infty} \int_K f(x_1, \dots, x_n, 1, 1, \dots)\,dP.$$

2.2. Estimate of the error of integration

Choose a grid consisting of points $X_1, \dots, X_N$ in K_∞. The projections of these points onto $K_{i_1\dots i_s}$ form an s-dimensional grid, whose discrepancy shall be denoted by $\Phi_\infty^{i_1,\dots,i_s}$.

Suppose that there exist constants B and $\{h_i\}$, $1 \leqslant i < \infty$, such that for any $i_1 < \dots < i_s$

$$\Phi_\infty^{i_1\dots i_s} \leqslant B h_{i_1} \cdots h_{i_s}. \tag{8}$$

In this case we shall say that the grid admits a multiplicative estimate of discrepancies.

Theorem. *For a grid $X_1, \dots, X_N$ admitting the multiplicative estimate of discrepancies (8), on classes of functions S_p and H_α whose defining constants decrease uniformly with respect to $\{h_i\}$, the estimate holds*

$$\left| \frac{1}{N} \sum_{\nu=1}^{N} f(X_\nu) - \int_{K_\infty} f(X)\,dX \right| \leqslant ABN^{-1/p} \exp \sum_{i=1}^{\infty} v_i, \tag{9}$$

where $v_i = \varepsilon_i h_i$ in the case of the class S_p, while in the case of the class H_α the parameter $p > 1/\alpha$ and

$$v_i = \varepsilon_i h_i (2^{1+\alpha} - 2^{1+1/p})^{-1}.$$

Proof scheme. Use inequality (3) for the function $f(x_1, \dots, x_n, 1, 1, \dots)$, then inequalities (5) or (7), (4), and (8). The right-hand side is estimated by means of Lemma 2 from (¹). Then pass to the limit as $n \to \infty$.

2.3. Use of the generalized Halton sequence

As the grid $X_1, \dots, X_N$ we choose the points $X_1^*, \dots, X_N^*$. Since the projections of the points $\{X_\nu^*\}$ onto $K_{i_1 \dots i_s}$ form an s-dimensional Halton sequence, it is easy to show (cf. (^{1},^{4})) that estimate (8) will be satisfied with $B = 1$, $h_i = \bar\beta_i \ln N + \bar\gamma_i$. The asymptotics of the coefficients as $i \to \infty$ is known: $\bar\beta_i \sim 4i$, $\bar\gamma_i \sim 8i \ln i$. Hence, for the applicability of the preceding theorem it is sufficient to require that the defining constants of the class decrease uniformly with respect to $\{i \ln i\}$.

Choose an arbitrary $\varepsilon > 0$. One can choose A and $\{\varepsilon_i\}$ so that $\sum \varepsilon_i \bar\beta_i = \varepsilon$; then the order of estimate (9) on the classes S_p will be $N^{-1/p+\varepsilon}$.

* According to (^{1}), this means that $L_{i_1 \dots i_s} \leqslant A\varepsilon_{i_1} \cdots \varepsilon_{i_s}$ and, in addition, the series $\sum_1^\infty \varepsilon_i h_i$ converges.

If p is fixed so that $0 < \alpha - 1/p < \varepsilon$, and then A and $\{\varepsilon_i\}$ are chosen so that $\sum \varepsilon_i \beta_i = (2^{1+\alpha} - 2^{1+1/p})(\varepsilon - \alpha + 1/p)$, then the order of the estimate (9) on the classes H_α will be $N^{-\alpha+\varepsilon}$.

We note that the orders of the estimates on the classes S_p and H_α, even in the n-dimensional case, cannot be better than $N^{-1/p}$ and, respectively, $N^{-\alpha}$.

2.4. Use of a generalized LP$_\tau$-sequence

Definition. By a generalized LP$_\tau$-sequence we shall mean a sequence of points $\{X_\nu^{**}\}$ with coordinates

$$X_{\nu+1}^{**} = (p^{(1)}(\nu),\ p^{(2)}(\nu), \dots,\ p^{(n)}(\nu), \dots),$$

where $\{p^{(n)}(\nu)\}$ is a sequence of type DR corresponding to the n-th monocyclic operator (5) (the operators are numbered so that their orders $\bar m_n$ do not decrease).

As the grid $X_1, \dots, X_N$ we choose the points $X_1^{**}, \dots, X_N^{**}$. Since the projections of the points $\{X_\nu^{**}\}$ onto $K_{i_1, \dots, i_s}$ form an s-dimensional LP$_\tau$-sequence, it follows from (5) that estimate (8) holds with $B = 1/2$, $h_i = 2^{\bar m_i}$, and also the asymptotic estimate for $\bar m_i$ as $i \to \infty$,

$$\bar m_i \le \log_2 i + \log_2(\log_2 i \ \log_2 \log_2 i) + O(1). \tag{10}$$

Thus, $h_i \le Ci \ln i \ln\ln i$, and for the applicability of our theorem it is sufficient to require that the defining constants of the class decrease uniformly with respect to $\{i \ln i \ln\ln i\}$.

The order of the estimate (9) on the classes S_p turns out to be $N^{-1/p}$. This is the best possible order.

Let us now consider the classes H_α. Let $0 < \varkappa < \varepsilon$. Fix $1/p = \alpha - \varkappa/\sqrt{\ln N}$. After a suitable choice of A and $\{\varepsilon_i\}$, the order of the estimate (9) will be equal to $N^{-\alpha+\varepsilon/\sqrt{\ln N}}$.

Remark. In this subsection we had to require a more rapid decrease of the defining constants in comparison with Sec. 2.3. Apparently this is caused by the crudeness of estimate (10).

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Note: Figure translations are in progress. See original paper for figures.

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