ON A GENERALIZATION OF PARABOLIC DIFFERENTIAL OPERATORS TO THE CASE OF MULTIDIMENSIONAL TIME

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Abstract

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ON A GENERALIZATION OF PARABOLIC DIFFERENTIAL OPERATORS TO THE CASE OF MULTIDIMENSIONAL TIME

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1. Let $P(\tau, \xi)$ be a polynomial in $\tau = (\tau_1, \dots, \tau_p) \in C^p$, $\xi = (\xi_1, \dots, \xi_n) \in C^n$, such that after replacing all τ_i by $(\tau_i)^{2b}$ it becomes a homogeneous polynomial of degree m ($2b$ is the weight of the variables τ_i). A polynomial P, homogeneous in the indicated sense, will be called **pluriparabolic** in the direction $N \in R^p$, if there exists an $a > 0$ such that $P(\tau + i\nu N, \xi) \neq 0$ for all $\tau \in R^p$, $\xi \in R^n$, $|\xi| = 1$, $\nu \geq -a$. Then $P(\tau, 0)$ is a homogeneous hyperbolic polynomial in the direction N of degree $k = m/2b$; $P(0, \xi)$ is a homogeneous elliptic polynomial of degree m. The differential operator $P(i\partial/\partial t, i\partial/\partial x)$ will also be called pluriparabolic. For $p = 1$ we obtain a parabolic operator in the sense of I. G. Petrovskii.

It is known that the hyperbolic polynomial $P(\tau, 0)$ will also be hyperbolic in any direction $M \in V$ (V is the convex cone that is the connected component of the set of points $\{\tau \in R^p;\ P(\tau, 0) \neq 0\}$ containing N). In other words, $P(\tau, 0) \neq 0$ when $\operatorname{Im}\tau \in V$. It turns out that the cone V plays an analogous role also for the polynomial $P(\tau, \xi)$.

Theorem 1. *The polynomial $P(\tau, \xi)$ is pluriparabolic in any direction $M \in V$. In other words, there exists a point $\omega \in V$ such that $P(\tau, \xi) \neq 0$ when $(\omega + \operatorname{Im}\tau) \in V;\ \xi \in R^n,\ |\xi| = 1$ (or $(\omega|\xi|^{2b} + \operatorname{Im}\tau) \in V$).*

Let $M \in V$; investigate the roots of $P(\tau + i\nu N + i\mu M, \xi)$ with respect to μ, for $\tau \in R^p$, $\xi \in R^n$, $\nu \geq -a$. Since the coefficient of the highest term in μ is distinct from zero ($P(M, 0) \neq 0$), the roots depend continuously on ν. By the pluriparabolicity of P there are no roots with $\operatorname{Re}\mu = 0$ when $\nu \geq -a$. By the hyperbolicity of $P(\tau, \xi)$ in τ (for fixed ξ), for sufficiently large ν there are no roots in the half-plane $\operatorname{Re}\mu > 0$; hence there are none there, by virtue of their continuous dependence on ν, for all $\nu \geq -a$.

2. It is convenient for us to study one class of operators that are correct in the sense of Petrovskii. A polynomial $P(\tau, \xi)$, whose principal part in τ, $P(\tau)$, does not depend on ξ (without the condition of homogeneity), is called **weakly pluriparabolic** in the direction N, if $P(N) \neq 0$ and $P(\tau, \xi)$ is

correct in the sense of Petrovskii with respect to N, i.e. $P(\tau+i\nu N,\xi)\neq 0$ for $\nu>a$ (a does not depend on ξ). Then, analogously to Theorem 1, it is shown that $P(\tau,\xi)\neq 0$ when $\operatorname{Im}\tau\in V+\omega$, where ω is a fixed point of R^p (not depending on ξ).

Theorem 2. *A weakly pluriparabolic operator $P(i\partial/\partial t, i\partial/\partial x)$ possesses a fundamental solution $G(t,x)$ such that $e^{(\omega,t)}G(t,x)\in S'(R^{n+p})$ and its support is contained in $\bar{V}^*\times R^n$ (where ω is the point appearing in the definition).*

The polynomial $P(\tau+i\omega+i\theta,\xi)$, for fixed $\theta\in V$, may be considered as an element of $S'(R^{n+p})$. Consider

$$G(\tau+i\omega,\xi)=\lim_{\substack{\theta\to 0\\ \theta\in V}}\frac{1}{P(\tau+i\omega+i\theta,\xi)},$$

where the limit is taken in the topology of $S'(R^{n+p})$.

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* V^* is the cone conjugate to the cone V: the set of such t that $(t,s)>0$ for all $s\in V,\ s\neq 0$; $\bar{V}$ is the closure of the cone V; $(t,s)=\sum t_j s_j$.

Taking the inverse Fourier transform, we obtain a function $G_\omega(t,x)\in S'(R^{n+p})$ with support in $\bar{V}^*\times R^n$, since $G(\tau+i\omega,\xi)$ was analytically continued in τ to the domain $\operatorname{Im}\tau\in V$. Multiplying $G_\omega(t,x)$ by $e^{-(\omega,t)}$, we obtain the required fundamental solution $G(t,x)$.

From the proof it is not difficult to see what conditions must be imposed on $G(t,x)$ in order that the operator $P(i\partial/\partial t, i\partial/\partial x)$ be weakly pluriparabolic: one must require, in addition, that the Fourier transform of $G(t,x)$ with respect to t admit analytic continuation to the domain $\operatorname{Im}\tau\in V+\omega$ in the form of a regular rapidly decreasing function of x.

Let us also note that if $G_\xi(t)$ is a fundamental solution of the hyperbolic operator $P(i\partial/\partial t,\xi)$ with support in $\bar{V}^*$, then $G(t,x)$ is the inverse Fourier transform of $G_\xi(t)$ with respect to ξ.

We shall consider $G(t,x)$ as a functional in $D'(R^{n+p})$. We shall say that a functional $f\in D'(R^{n+p})$ is spatially bounded if its support in t (the union over all ξ) has compact intersections with all translates of the cone $-V^*$. It is obvious that the convolution of any such functional, finite with respect to x, exists and is a solution of the equation $P(i\partial/\partial t, i\partial/\partial x)u=f$.

These considerations can be applied in solving the Cauchy problem. Consider a hyperplane $\Omega\subset R^p$ noncharacteristic with respect to $P(i\partial/\partial t,0)$ (its normal N belongs to V). Replace the variables t_i by $(r,s_i,\dots,s_{p-1})$ so that the equation of Ω is $r=0$, and on the normal $s_i=\text{const}$; let $(\rho,\sigma_1,\dots,\sigma_{p-1})$ be the dual variables. Put $\Sigma=\Omega\times R^n$. On Σ we shall prescribe Cauchy data. This yields a characteristic Cauchy problem: the number of data is equal to k (the degree of $P(\tau,0)$). Let R^+ be the closed half-space in R^{n+p}, bounded by Σ and containing

N. If a generalized function $u \in D'(R^{n+p})$ with support in R^+ is such that the support of $h = P(i\partial/\partial t, i\partial/\partial x)u$ is contained in Σ, then a direct verification shows that h is expressed only through the limits of the derivatives of u with respect to r as $r \to 0$: $u(0,s;x), u^{(1)}(0,s;x), \ldots, u^{(k-1)}(0,s;x)$; $u = h * G$ (h is a spatially bounded functional). Thus, if $u^{(i)}(0,s;x)$ are prescribed, then, constructing h from them, we obtain a solution of the Cauchy problem with zero right-hand side; here the initial data must be such that the convolution exists. If one solves the Cauchy problem for ordinary functions, then one can use the known results on equations correct in the sense of Petrovskii ([2]); in our case, moreover, no restrictions on growth need be imposed with respect to the variables t.

The general case of the Cauchy problem is reduced in a known way to the case $u^{(i)} = 0$ for $i < k-1$, $u^{(k-1)}(0,s;x) = v(s,x)$. In this case, under a suitable normalization of the coordinates, $h = \delta(r)v$ and $u = G * v$ (the convolution is taken with respect to s and x). An even simpler case, to which the general one also reduces, is $v = \delta(s)w(x)$. Then h is a functional concentrated in the n-dimensional plane $t = 0$; $u = G(t,x) * w(x)$, and the functional $u(t,x)$ is concentrated in $\bar{V}^* \times R^n$. Hence, in particular, it follows that the solution does not depend on the choice of the plane Ω. Thus, if one requires that the support of $h = P(i\partial/\partial t, i\partial/\partial x)u$ be concentrated in the plane $t = 0$, then to determine u one must prescribe k functions: $u^{(i)}(0,x)$, $0 \le i \le k-1$, derivatives in some direction contained in V (it does not matter which one). A Cauchy problem arises with data on a plane of codimension k (a multidimensional Cauchy problem).

We now study in more detail the fundamental solution $G(t,x)$ for a pluriparabolic operator. Let $E(r,\sigma,\xi)$ be its Fourier transform with respect to s and x. Then $E(r,\sigma,\xi)$ is an entire function of σ and ξ, admitting the estimates:

$$|E(r,z,w)| \le c_1(1+|z|+|w|)^l \exp(r(b|z| + c|w|^{2b})),$$

$$r > 0, \quad z \in C^{p-1}, \quad w \in C^n;$$

$$|E(r,\sigma,\xi)| \le c_2(1+|\sigma|+|\xi|)^l \exp(-ar|\xi|^{2b}),$$

$$r > 0, \quad \sigma \in R^{p-1}, \quad \xi \in R^n, \quad a > 0.$$

These estimates are obtained in the standard way from consideration of the roots $P(\rho,\sigma;\xi)$ with respect to ρ. Then, from the general facts on the Fourier transform of entire functions (2), it follows that $G(r,s;x)$ is a finite functional in s with support independent of x (this is equivalent to saying that the support of $G(t,x)$ with respect to t is contained in $\overline{V}^*$), and the Fourier transform $\widehat{G}(r,z;x)$

of $G(r, s; x)$ with respect to s is a regular function; moreover, for every z, for arbitrarily large $d > 0$, the estimate holds

$$|\widehat{G}(r, z; x)| \leq c \exp(-d|x|^{2b'}),$$

where $1/2b + 1/2b' = 1$ for sufficiently small r. Hence it follows

Theorem 3. *If, for a pluriparabolic operator, the Cauchy data are sufficiently smooth in s and, in every domain D bounded with respect to s, satisfy the estimate*

$$|u^{(j)}(0, s; x)| \leq c \exp(b(D)|x|^{2b'}),$$

then there exists a unique solution which, for any $\varepsilon > 0$ and any domain $D' \subset D$, for sufficiently small s, satisfies the estimate

$$|u^{(j)}(r, s; x)| \leq c \exp((b(D) + \varepsilon)|x|^{2b'}).$$

Theorem 3 shows that, in questions of uniqueness and correctness of the Cauchy problem, pluriparabolic operators behave like hyperbolic ones with respect to the variables t, and like parabolic ones with respect to the variables x.

3. We now consider one special class of pluriparabolic operators. Let $P(\partial/\partial t)$, $t \in R^p$, be a homogeneous hyperbolic operator; $P(\tau) \neq 0$ for $\operatorname{Im} \tau \in V$. Let, further, $F(\xi) = \{F_1(\xi), \dots, F_p(\xi)\}$ be a mapping of the space R^n into C^p, all components of which $F_j(\xi)$ are homogeneous polynomials of degree $2b$. We shall call the mapping $F(\xi)$ V-elliptic if $\operatorname{Im} F(\xi) \in V$ for $\xi \in R^n$, $|\xi| \neq 0$.

Theorem 4. *For every homogeneous hyperbolic operator $P(\partial/\partial t)$ and every V-elliptic mapping $F(\xi)$, the operator*

$$P(\partial/\partial t + F(i\partial/\partial x)) \tag{1}$$

is pluriparabolic.

It is enough to observe that the values $\operatorname{Im} F(\xi)$ for $|\xi| = 1$ form a compact set in V, which can be enclosed in a translate of the cone V to some point $\omega \in V$.

In the case $p = 1$ and $P(\partial/\partial t) = \partial/\partial t$, the operator (1) becomes a parabolic operator of the simplest form: $\partial/\partial t + F(i\partial/\partial x)$; $F(i\partial/\partial x)$ in this case is a semibounded homogeneous elliptic operator in the usual sense. For operators (1) the fundamental solution takes a particularly simple form.

Theorem 5. *The fundamental solution $G(t, x)$ of the operator (1) has the form:*

$$G(t, x) = G(t)\psi(t, x),$$

where $G(t)$ is the fundamental solution of the operator $P(\partial/\partial t)$ with support in $\overline{V}^*$; $\psi(t,x)$ is the inverse Fourier transform (with respect to ξ) of the function $\exp(-(t,F(\xi)))$, *$t\in V^*$. Hence, first, it follows that if the vector $N\in\overline{V}^*$, then $G(\nu N,x)=g_N(\nu,x)$, $\nu>0$, where $g_N(\nu,x)$ is the fundamental solution of the parabolic equation $\partial/\partial\nu+(N,F(i\partial/\partial x))=0$. Secondly,*

$$\psi(t,x)*\psi(s,x)=\psi(t+s,x),\qquad t,s\in\overline{V}^*, \tag{2}$$

where the convolution is taken with respect to x. Finally, the support of $G(t,x)$ is the direct product of the support of $G(t)$ and R^n. In particular, if there are lacunae for $G(t)$, or if the Huygens principle holds for $P(\partial/\partial t)$, then the same holds for the operator (1).

4. From (2) it follows that with the operator (1) one can associate a p-parameter semigroup of operators $T(t)\varphi(x)=\psi(t,x)*\varphi(x)$. These operators, up to multiplication by $G(t)$, give the solution of the multidimensional problem-

Cauchy problem. This is an analogue of the known facts on the connection of parabolic operators with one-parameter semigroups. Further, if the degree of all $F_j(\xi)$ is equal to 2, then in this (and only in this) case $\psi(t,x)\geq 0$. Moreover, always

$$\int_{R^n}\psi(t,x)\,dx=1,$$

so that in this case $\psi(t,x)$ can be interpreted as the density of transition probabilities: the probability of transition from the point (t_0,x_0), $t_0\in R^p$, $x_0\in R^n$, to the domain (t_0+t,E), $t\in\overline{V}^*$, $E\subset R^n$, is equal to

$$\int_E\psi(t,x-x_0)\,dx.$$

Then (2) is an analogue of the Chapman–Kolmogorov equation, and (1) is an analogue of the Fokker–Planck equation. If there is a system of transition probabilities satisfying the listed requirements in an arbitrary cone V, then one can always construct an analogous equation (1) in convolutions (but not always a differential one).

On the other hand, again starting from (2), it is natural to consider an abstract analogue of the operators (1). Let $A_1,\dots,A_p$ be a set of commuting operators in a Banach space E. Consider the equation

$$P(\partial/\partial t+A)u=0,\qquad u\in E,\qquad t\in R^p. \tag{3}$$

We shall solve for (3) the multidimensional Cauchy problem of the simplest form $u^{(j)}(0,s)=0$ for $j<k-1$, $u^{(k-1)}(0,s)=\delta(s)u_0$. The solution of such a problem

is equivalent to the construction of a p-parameter semigroup $T(t) = \exp((t, A))$, $u = G(t)T(t)u_0$. If the operators A_i are bounded and we are interested in bounded solutions $u(t)$, $t \in \overline{V}^*$, then it is sufficient to require that the joint spectrum of the operators A_i lie in the domain $(-\operatorname{Re}\lambda) \in V^*$ (the condition $(-\operatorname{Re}\lambda) \in \overline{V}^*$ is necessary). If the operators A_i are unbounded, but have a dense domain of definition and, for every $N \in \overline{V}^*$, the estimate

$$\left\|(\lambda - (N, A))^{-1}\right\| \leq 1/\lambda$$

holds for $\lambda > 0$, then there exists a strongly continuous contraction semigroup $T(t)$, $t \in \overline{V}^*$, giving the solution of the Cauchy problem. The operators A_i are naturally interpreted as the infinitesimal operators of the semigroup $T(t)$. They are defined for any strongly continuous semigroup $T(t)$; moreover, for a contraction semigroup they satisfy the estimate just given. If A_i are normal operators in a Hilbert space, then they generate a semigroup with the indicated properties if and only if their joint spectrum lies in the domain $(-\operatorname{Re}\lambda) \in \overline{V}^*$; in this case

$$\exp((t, A)) = \int_\Lambda \exp((t, \lambda))\, dE_\lambda,$$

where E_λ is the joint spectral family of the A_i, and Λ is the joint spectrum.

Returning again to differential operators, we note that if $\operatorname{Re} F(\xi) \in \overline{V}^*$, then the operator (1) will be weakly pluriparabolic. For arbitrary $F(\xi)$, one can, following (3), solve a well-posed problem by taking as initial data functions whose Fourier transforms (in x) have supports belonging to the set where $\operatorname{Re} F(\xi) \in \overline{V}$.

5. A polynomial $P(\xi)$, $\xi \in R^n$, is called **strongly homogeneous** if a group of affine transformations g of the space R^n such that $P(g\xi) = a(g)P(\xi)$ acts transitively on the connected component $D(P)$ of the set $\{\xi : P(\xi) \neq 0\}$. In [4] it is proved that every strongly homogeneous polynomial $P(\xi)$ for which $D(P)$ contains no straight lines is weakly pluriparabolic, reducible by means of affine transformations to the form (1), where all components $F_j(\xi)$ have degree 2.

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Note: Figure translations are in progress. See original paper for figures.