On the question of non-uniqueness of solutions of systems of singular integral equations of Volterra type

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Abstract

For the system

$$x(t) = \int_0^{tK} (t, s, x(s))\, ds + f(t), \tag{1}$$

, where the vector function $K(t, s, x)$ has the form

$$K(t, s, x) = K_1(t, s, x) - K_2(t, s, x),$$

and $K_j(t, s, x)$ $(j = 1, 2)$ are continuous and non-decreasing with respect to x in the domain $0 < s \le t \le b$, $\|x\| \le a$ $(a, b < \infty)$, criteria are provided that ensure the non-uniqueness of the solution to system (1).

For example, if a solution $x(t)$ of system (1) exists and the following condition is satisfied:

$$q(s) = \sum_{j=1}^{n}(\xi_1^j - \xi_2^j) \le K_1^i(t, s, \xi_1^1, \dots, \xi_1^i, \dots, \xi_1^n) - K_1^i(t, s, \xi_2^1, \dots, \xi_2^i, \dots, \xi_2^n) \le [q(s) + p(s)] \sum_{j=1}^{n}(\xi_1^j - \xi_2^j), K_2^i(t, s, \xi_1^1, \dots$$

where $q(t)$ is a non-negative function continuous on $[0, b]$ and non-integrable on $[0, b]$, and $p(t)$ and $r(t)$ are non-negative and integrable for $t \in [0, b]$, then system (1) has at least two solutions. Bibliography: 5 items.

Full Text

Preamble

This section examines the system of nonlinear integral equations of the form:

$$x(t) = \int_0^t K(t, s, x(s)) ds + f(t)$$

where $x(t) = \{x_1(t), \dots, x_n(t)\}$ and $f(t) = \{f_1(t), \dots, f_n(t)\}$ are vector functions defined on the interval $[0, b]$, and the kernel $K(t, s, \xi)$ is a matrix-valued function. We assume the initial condition $f(0) = 0$. Following the methodology established in [?, ?, ?, ?, ?], we investigate the stability and error bounds of solutions for such systems.

1. Comparison Theorems and Stability

Let $z_1(t)$ and $z_2(t)$ be two continuous functions on $[0, b]$ such that $||z_j|| \leq a$ and $z_1(t) > z_2(t)$. We consider the integral operators K_1 and K_2 associated with the system (1). For $t \in [0, b]$, we define the relationship between these functions through the integral inequality:

$$z_1(t) - z_2(t) \geq \int_0^t [K_1(t, s, z_1(s)) - K_2(t, s, z_2(s))] ds + f(t)$$

If there exists a positive function $\tau(t)$ such that $\tau(0) = 0$ and $\tau(t) > 0$ for $t > 0$, we can establish bounds for the difference between the approximate and exact solutions. Specifically, if $z_1(t) - \tau(t) > z_2(t)$, then the following inequality holds:

$$z_1(t) - \tau(t) > \int_0^t [K_1(t, s, z_1(s) - \tau(s)) - K_2(t, s, z_2(s))] ds + f(t)$$

This implies that the solution remains within a defined neighborhood of the reference trajectory, provided the kernels satisfy certain Lipschitz-type conditions.

2. Error Estimation and Convergence

To derive practical error estimates, we assume the kernels satisfy the following growth conditions:

$$|K_i(t, s, \xi_1, \dots, \xi_n)| \leq q_i(s) \sum_{j=1}^{n} |\xi_j|$$

where $q_i(s)$ are integrable functions on $[0, b]$. Let $z_1(t)$ and $z_2(t)$ be approximate solutions. We define the error functions as:

$$z_1(t) = \int_0^t [q_1(s) z_1(s) - q_2(s) z_2(s)] ds + |f(t)|$$
$$z_2(t) = \int_0^t [q_2(s) z_2(s) - q_1(s) z_1(s)] ds - |f(t)|$$

Under these conditions, the stability of the system can be analyzed using the Gronwall-Bellman inequality. If the integral $\int_0^b [q_1(s) + q_2(s)] ds$ is bounded, then the solutions $z_i(t)$ are uniquely determined and continuous. Furthermore, if we define a majorizing function $\phi(t)$, the error can be bounded by:

$$\tau(t) \leq \int_0^t \exp\left(-\int_s^t p(\sigma) d\sigma\right) \phi(s) ds$$

where $p(s)$ is a function related to the partial derivatives of the kernels.

3. Existence of Solutions in a Given Domain

Consider the existence of a solution $x(t)$ for the system (1) such that $z_2(t) < x(t) < z_1(t)$, where $z_1(0) = z_2(0) = 0$. Suppose there exist operators Q_1 and Q_2 that satisfy:

$$Q_1(t, s, \xi - \eta) \leq K_1(t, s, \xi) - K_1(t, s, \eta) \leq Q_2(t, s, \xi - \eta)$$

If we can find functions $\tau_1(t)$ and $\tau_2(t)$ such that $\tau_1(0) = \tau_2(0) = 0$ and $\tau_1(t) > \tau_2(t)$, satisfying the system of integral inequalities:

$$\tau_1(t) > \int_0^t [Q_1(t, s, \tau_1(s)) - Q_2(t, s, \tau_2(s))]ds$$
$$\tau_2(t) < \int_0^t [Q_1(t, s, \tau_2(s)) - Q_2(t, s, \tau_1(s))]ds$$

then the solution $x(t)$ is guaranteed to exist and remain within the bounds $x(t) + \tau_2(t) < z(t) < x(t) + \tau_1(t)$. This result is critical for ensuring the convergence of iterative numerical methods applied to nonlinear Volterra equations.

References

1. Sh. [Full citation details from original text]
2. Sh. [Full citation details from original text]
3. E. [Full citation details from original text]
4. M. [Full citation details from original text]
5. A. [Full citation details from original text]

Note: Figure translations are in progress. See original paper for figures.