# ON THE PERTURBATION OF HOMOMORPHISMS BY OPERATORS OF FINITE RANK

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## Abstract

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# ON THE PERTURBATION OF HOMOMORPHISMS BY OPERATORS OF FINITE RANK

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Many authors have considered, in Banach spaces, closed linear operators with closed range for which at least one of the numbers $\alpha$ (the dimension of the null space of the operator) or $\beta$ (the codimension of the range of the operator) is finite. For operators with such properties it has been proved (see ([1]) and the bibliography there) that linear completely continuous perturbations do not destroy these properties. However, if $\alpha$ and $\beta$ are both infinite, then one can always indicate a linear completely continuous operator of infinite rank (even arbitrarily small in norm) whose addition destroys the closedness of the range ([2]). In this connection there arises the question of the preservation of the closedness of the range of a closed linear operator with infinite $\alpha$ and $\beta$ under its perturbation by linear continuous operators of finite rank. This question has been answered affirmatively for operators in Hilbert space ([3], [4]).

In the present note an analogous question is studied for linear topological spaces; moreover, instead of closed operators, homomorphisms are considered, i.e., linear operators that map sets open with respect to the domain onto sets open with respect to the range. We note that in Banach spaces closed operators with closed range are homomorphisms (([5]), Theorem 11), but homomorphisms with closed range need not be closed operators.

**Lemma 1.** *Let $E$ be a linear topological space (l.t.s.); $L$ its subspace (i.e., a linear set closed in $E$); $M$ a linear set in $L$. Then the image of the subspace $L$ under the canonical homomorphism $\varphi$ of the space $E$ onto $E/M$ is closed in $E/M$.*

**Proof.** Since $E \setminus L$ is open in $E$, $\varphi(E \setminus L)$ is open in $E/M$. Since $M \subseteq L$, we shall have $\varphi(L) \cap \varphi(E \setminus L) = \emptyset$. Hence we conclude that $\varphi(L)$ is closed in $E/M$, for $\varphi(L) \cup \varphi(E \setminus L) = E/M$.

**Lemma 2.** *Let $E$ and $F$ be l.t.s.; $A$ a homomorphism of $E$ into $F$; $Z_A$ the kernel of the homomorphism $A$. If $L$ is a subspace in $E$ containing $Z_A$, then $A(L)$ is closed in $A(E)$.*

**Proof.** Consider the operator $\widetilde{A}$, defined on $E/Z_A$ by the equality $\widetilde{A}\tilde{x} = Ax$, where $x$ is any element of the equivalence class $\tilde{x}$. For any set $G$ in $E/Z_A$ we have $\widetilde{A}(G) = A(\varphi^{-1}(G))$, where $\varphi$ is the canonical homomorphism of $E$ onto $E/Z_A$. If $G$ is open in $E/Z_A$, then $\varphi^{-1}(G)$ is open in $E$ and, consequently, $\widetilde{A}(G)$ is open in $A(E)$. Thus $\widetilde{A}$ is a homomorphism. Hence, taking into account the one-to-one character of $\widetilde{A}$, we conclude that $\widetilde{A}$ maps every closed set in $E/Z_A$ onto a closed set in $\widetilde{A}(E/Z_A) = A(E)$. Since $A = \widetilde{A} \circ \varphi$, we have $A(L) = \widetilde{A}(\varphi(L))$, whence, on the basis of Lemma 1, we conclude that $A(L)$ is closed in $A(E)$.

**Theorem 1.** *Let $E$ and $F$ be l.t.s.; $A$ a homomorphism of $E$ into $F$; $K$ a linear closed operator of finite rank. If $A(E)$ is closed in $F$, then $(A+K)(E)$ is also closed in $F$.*

**Proof.** First of all, let us note that from the closedness of $K$ follows the closedness of $Z_k$ in $E$. Put $M_1 = Z_k$, $M_2 = Z_A \cap (E \ominus Z_k)$,

$$M_3 = (E \oplus Z_k) \oplus M_2$$

(the symbol $\oplus$ is used to denote algebraic complements, which are regarded here as fixed in some way; moreover, the complement $E \oplus Z_k$ occurring in the expressions for $M_2$ and $M_3$ denotes one and the same set). By the condition of the theorem, the space $E \oplus Z_k$ is finite-dimensional. Therefore the set $L = M_1 \oplus M_2$ is closed in $E$ ((6), Ch. I, § 2, no. 3, Corollary 4). Hence, since $Z_A \subseteq L$, on the basis of Lemma 2 we conclude that $A(L)$ is closed in $A(E)$. But $A(E) = \overline{A(E)}$, and hence $A(L)$ is closed in $F$. Further, we have

$$(A+K)(E) = (A+K)(L) + (A+K)(M_3) = A(L) + K(M_2) + (A+K)(M_3),$$

where $K(M_2)$ and $(A+K)(M_3)$ are finite-dimensional spaces. Consequently, $(A+K)(E)$ is closed in $F$.

**Lemma 3.** In an l.t.s. $E$, every projection operator $P$ (i.e., a linear operator mapping $E$ into itself and satisfying the condition $P^2 = P$) is a homomorphism.

**Proof.** Let $G$ be an open set in $E$, and let $P(E) = L$. It is easy to see that

$$P(G) = L \cap \left( \bigcup_{x \in G} (G - x + Px) \right).$$

Since

$$\bigcup_{x \in G} (G - x + Px)$$

is open in $E$, $P(G)$ is open in $L$.

Let us note that if the projection operator $P$ is continuous, then the openness of $P(G)$ in $P(E)$ follows from Propositions 4 ((7), Ch. I, § 8, no. 4) and 11 ((6), Ch. I, § 1, no. 8).

**Corollary.** Let $A$ be a homomorphism from an l.t.s. $E$ to an l.t.s. $F$; let $E_1$ and $E_2$ be linear sets in $E$ such that

$$A(E_1) \cap A(E_2) = \{0\}$$

and

$$A(E_1) + A(E_2) = A(E).$$

If $G$ is open in $E$, $G_i \subseteq E_i$ $(i = 1, 2)$, and

$$A(G) = A(G_1) + A(G_2),$$

then $A(G_i)$ is open in $A(E_i)$ $(i = 1, 2)$.

Indeed, the set $A(G)$ is open in $A(E)$ by hypothesis. The set $A(G_i)$ $(i = 1, 2)$ is the projection of $A(G)$ onto $A(E_i)$. Consequently, by Lemma 3, $A(G_i)$ is open in $A(E_i)$.

**Lemma 4.** Let $E$ and $F$ be separated l.t.s. Every linear operator $K$ of finite rank mapping $E$ into $F$ is a homomorphism.

**Proof.** Denote by $M$ some complement to $Z_k$ in $E$ (clearly, $M$ is finite-dimensional). Let $G$ be an open set in $E$; let $G'$ be the projection of $G$ onto $M$ (under the projection generated by the decomposition $E = Z_k \oplus M$). By Lemma 3, the set $G'$ is open in $M$. Restricting $K$ to $M$, we obtain an invertible operator on the finite-dimensional space $M$ with range $K(M) = K(E)$. Since a linear operator defined on a separated finite-dimensional l.t.s. is continuous, the set $K(G')$ $(= K(G))$ is open in $K(M)$, i.e., $K(G)$ is open in $K(E)$.

**Lemma 5.** Let $E_i$ $(1 \leq i \leq n)$ be disjoint subspaces of an l.t.s. $E$;

$$\oplus \sum_{i=1}^{n} E_i = E;$$

let $P_i$ $(1 \leq i \leq n)$ be the projection operators of $E$ onto $E_i$, generated by the decomposition

$$E = \oplus \sum_{i=1}^{n} E_i.$$

If all $P_i$ are continuous, then the sets of the form

$$\sum_{i=1}^{n} G_i,$$

where $G_i \subseteq E_i$ is an arbitrary neighborhood of zero open in $E_i$, are open in $E$ and form in $E$ a base of neighborhoods of zero.

**Proof.** Since the operator $P_i$ is continuous, the full preimage

$$P_i^{-1}(G_i) \left( = G_i + \sum_{k \neq i} E_k \right)$$

of the set $G_i$ is open in $E$. It is not difficult to verify that

$$\sum_{i=1}^{n} G_i = \bigcap_{i=1}^{n} \left( G_i + \sum_{k \neq i} E_k \right),$$

whence the openness of

$$\sum_{i=1}^{n} G_i$$

in $E$ follows.

Let $U$ be an arbitrary neighborhood of zero. There exist such (open) neighborhoods of zero $U_i$ $(1 \leq i \leq n)$ that

$$\sum_{i=1}^{n} U_i \subseteq U.$$

Putting

$$G_i = U_i \cap E_i,$$

we shall have

$$\sum_{i=1}^{n} G_i \subseteq U,$$

and $G_i$ is open in $E_i$.

**Lemma 6.** Let $F_i$ $(1 \leq i \leq n)$ be subspaces of a separable l.t.s. $F$;

$$\sum_{i=1}^{n} F_i = F;$$

$H_i$ are open sets in $F_i$. If the subspaces $F_2, \dots, F_n$ are finite-dimensional, then the sum

$$\sum_{i=1}^{n} H_i$$

is open in $F$.

**Proof.** We first consider the case when $F_i \cap F_j = \{0\}$ for $i \neq j$. By virtue of the separability of $F$ and the finite-dimensionality of $F_i$ $(2 \leq i \leq n)$, the projection operators corresponding to the decomposition

$$F = \oplus \sum_{i=1}^{n} F_i$$

are continuous (([6], Ch. I, § 2, no. 3, Proposition 3, and Ch. 1, § 1, no. 8, Proposition 12). Therefore Lemma 5 is applicable, according to which the sum

$$\sum_{i=1}^{n} H_i$$

is open in $F$.

Let us consider the case when $F_i \cap F_j \neq \{0\}$ for some $i, j$ $(i \neq j)$. Without loss of generality in the proof, we may take $n = 2$. In $F_2$ take a complement $\widetilde{F}$ to $F_1 \cap F_2$ (so that $F = F_1 \oplus \widetilde{F}$, $\widetilde{F} \subset F_2$). Let $P$ and $Q$ be the projection operators of the space $F$ onto $F_1$ and $\widetilde{F}$, generated by the decomposition $F = F_1 \oplus \widetilde{F}$. Take $x \in H_1$, $y \in H_2$. It is required to show that $H_1 + H_2$ contains a neighborhood $U$ (in $F$) of the point $x + y$. Put $\widetilde{H} = (H_2 - Py) \cap \widetilde{F}$. The set $\widetilde{H}$ is open in $\widetilde{F}$ and contains the point $Qy$. Since $F_1 \cap \widetilde{F} = \{0\}$, from the first part of the proof we conclude that $H_1 + \widetilde{H}$ is open in $F$. The set $U = H_1 + \widetilde{H} + Py$ is also open in $F$, is contained in $H_1 + H_2$, and contains $x + y$.

**Theorem 2.** Let $E$ and $F$ be separable l.t.s.; $A$ a mapping of $E$ into $F$ with closed range; $K$ a linear continuous operator from $E$ into $F$ of finite rank. If $A$ is a homomorphism of $E$ into $F$, then $A + K$ is also a homomorphism of $E$ into $F$.

**Proof.** Take some open set $G$ in $E$. It is required to prove that $(A + K)(G)$ is open in $(A + K)(E)$. In the notation of Theorem 1 we have

$$E = \oplus \sum_{i=1}^{3} M_i.$$

By Lemma 5, the set $G$ can be represented in the form

$$G = \bigcup_{x \in G} G_x,$$

where $G_x$ is an open neighborhood of the point $x$, equal to the sum of its projections onto the subspaces $M_i$. Since

$$(A + K)(G) = \bigcup_{x \in G} (A + K)(G_x),$$

to prove the theorem it is enough to establish the openness in $(A + K)(E)$ of the sets $(A + K)(G_x)$, $x \in G$. Without loss of generality, we may assume that the set $G$ itself is equal to the sum of its projections onto $M_i$:

$$G = \sum_{i=1}^{3} G_i.$$

Since

$$(A + K)(E) = A(M_1) + K(M_2) + (A + K)(M_3),$$

$$(A + K)(G) = A(G_1) + K(G_2) + (A + K)(G_3),$$

the required result follows from Lemma 6 if we put $F_1 = A(M_1)$, $F_2 = K(M_2)$, $F_3 = (A + K)(M_3)$, $H_1 = A(G_1)$, $H_2 = K(G_2)$, $H_3 = (A + K)(G_3)$. It remains only to show that $F_i$ are closed in $F$ and $H_i$ are open in $F_i$ ($i = 1, 2, 3$). For $i = 2, 3$ this follows from Lemma 4, taking into account the finite-dimensionality of $M_2$ and $M_3$. Consider $i = 1$. Since $A(E) = A(M_1)+A(M_3)$, $A(M_1)\cap A(M_3) = \{0\}$, and $A(G) = A(G_1) + A(G_3)$, it follows from Lemma 3 that $A(G_1)$ is open in $A(M_1)$ (i.e. $H_1$ is open in $F_1$). As for the closedness of $A(M_1)$ in $F$, it was established in the proof of Theorem 1 (there it was shown that $A(L)$ is closed in $F$, where $A(L) = A(M_1 \oplus M_2) = A(M_1)$).

We note that the closedness of the operator $K$ in Theorem 1 and its continuity in Theorem 2 were needed only in order to conclude that $Z_k$ is closed in $E$. In Theorem 2, since $F$ is separable and the operator $K$ is defined on all of $E$, the conditions of continuity and closedness of $K$ are equivalent.

In conclusion we shall show that the requirements that the operator $A$ be homomorphic and that its range be closed are both essential for each of Theorems 1 and 2. To this end consider the following example. Let $E$ be a Banach space, and let $L$ be a linear nonclosed set in $E$ having a one-dimensional complement $L_1$ in $E$ ($L$ may be regarded as the kernel of a linear unbounded functional on $E$). Denote by $P$ the projection operator of $E$ onto $L$, generated by the decomposition $E = L \oplus L_1$. By Lemma 3 the operator $P$ is a homomorphism. Take $x_1 \in L_1$, construct a bounded linear functional $f$ such that $f(x_1) = 1$, and put $Kx = f(x)x_1$, $x \in E$, $A = P + K$. We shall show that $A(E) = E$, i.e. that the equation $Px + f(x)x_1 = y$ is solvable for every $y \in E$. Let $y = y_1 + \alpha x_1$, where $y_1 = Py$, and $\alpha$ is the number uniquely determined by the element $y$. The equation $Px = y_1$ has solutions of the form $y_1 + \beta x_1$, where $\beta$ is an arbitrary number. It is required to choose $\beta$ so that the equality $P(y_1 + \beta x_1) + f(y_1 + \beta x_1)x_1 = y_1 + \alpha x_1$ be satisfied, i.e. $[f(y_1) + \beta]x_1 = \alpha x_1$. Hence $\beta = \alpha - f(y_1)$. Thus it has been shown that $A$ has a closed range. By Theorem 2, $A$ cannot be a homomorphism (for otherwise the operator $P = A - K$ would have a closed range). Thus, the operators $A$ and $P$ each satisfy only one of the two conditions considered in Theorems 1 and 2, although these operators differ from one another only by the continuous finite-rank operator $K$.

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*Note: Figure translations are in progress. See original paper for figures.*

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