Reports of the Academy of Sciences of the USSR

MATHEMATICAL PHYSICS

1967

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1967. Volume 173, No. 2

UDC 513.88 : 517.537

MATHEMATICAL PHYSICS

D. Ya. PETRINA

ON THE COMPLETENESS OF THE AMPLITUDES OF PERTURBATION THEORY IN THE SPACE OF AMPLITUDES

(Presented by Academician N. N. Bogolyubov on 4 V 1966)

1. Lehmann ([1]) established that the scattering amplitudes for processes $1+2 \to 3+4$ are holomorphic functions of the cosine of the scattering angle inside a certain ellipse. It is therefore expedient to introduce the following definition.

A linear space consisting of functions of the complex variable z, holomorphic inside an ellipse E with foci at the points $(-1, 0)$, $(1, 0)$ and with major semiaxis $\eta > 1$, will be called the **space of amplitudes** and denoted by H_0 (the number η is determined according to Lehmann' s theorem).

For some models, for example nonrelativistic scattering by the Yukawa potential, $V(r) = e^{-\mu r}/r$, $\mu = 1$, and relativistic scattering described by the Bethe-Salpeter equation in the ladder approximation, the scattering amplitudes, for a sufficiently small value of the interaction constant g, can be represented in the form

$$f(s, \cos\theta, g) = \sum_{n=0}^{\infty} g^{n+1} f^n(s, \cos\theta), \tag{1}$$

where the functions $f^n(s, \cos\theta)$ are holomorphic in $\cos\theta$ inside the corresponding Lehmann ellipses, and the series (1), for fixed s and $\cos\theta$ from a certain domain, converges uniformly in g in some neighborhood of zero. The functions $f^n(s, \cos\theta) = f^n(s, z)$ will be called **amplitudes of perturbation theory**. In the relativistic case (the Bethe-Salpeter equation), the functions $f^n(s, z)$ are contributions from ladder Feynman diagrams (with equal internal and external masses $\mu = 1$).

In the present note the functions $f^n(s, z)$ are considered for fixed $s > 0$ in the case of nonrelativistic scattering and for $3 < s < 4$ in the case of relativistic scattering. Under these restrictions on s, the functions $f^n(s, z)$ are holomorphic inside ellipses E_n with major semiaxis equal to $\operatorname{ch}(n+1)\alpha$, $\operatorname{ch}\alpha = 1+1/2s = \eta$, in the nonrelativistic case, and with major semiaxis equal to $|1+2(n+1)^2/(s-4)|$, $\operatorname{ch}\alpha = 1+2/(s-4)$, in the relativistic case.

2. Introduce the functions

$$\tilde{f}^n(s,z) = \frac{1}{M_n L_n} f^n(s,z),$$

where $M_n = \max|f^n(s,z)|$ in the ellipse with major semiaxis equal to $\operatorname{ch}(n+1)(\alpha-\varepsilon)$ in the nonrelativistic case and to $|1+2(n+1)^2/(s-4)|-\varepsilon$ in the relativistic case; $\varepsilon > 0$ is a sufficiently small number, and L_n is the length of the ellipse.

Consider, in addition, the functions

$$f_l(s,g) = \frac{1}{2}\int_{-1}^{1} P_l(z) f(s,z,g)\,dz,$$

where $P_l(z)$ is a Legendre polynomial. These functions are meromorphic in g in the complex plane and, as l increases, decrease exponentially, $f_l(s,g) \sim e^{-\alpha l}$ ([2,3]).

The following two lemmas are valid; they characterize the strengthened linear independence of the functions $\tilde{f}^n(s,z)$ and $f_l(s,g)$.

Lemma 1. The expression $\sum_{n=0}^{\infty} a_n \tilde{f}^n(s,z)$, where $\sum_{n=0}^{\infty}|a_n|^2 < \infty$, can be equal to zero for $-1 \le z \le 1$ only in the case when all $a_n = 0$.

The proof of this lemma follows from the fact that the functions $f^n(s,z)$ are holomorphic in ever-increasing domains, and their singularities cannot compensate one another.

Lemma 2. The expression

$$\sum_{l=0}^{\infty} a_l \sqrt{2(2l+1)}\, f_l(s,g), \quad \text{where } \sum_{l=0}^{\infty} |a_l|^2 < \infty$$

can be equal to zero for all g only in the case when all $a_l = 0$ (for almost all s from the interval $s > 0$ in the relativistic case and $3 < s < 4$ in the nonrelativistic case—only for such s are all subsequent results valid).

The proof of Lemma 2 is carried out according to the same scheme as the proof of Lemma 1; here one interesting property of the functions $f_l(s,y)$ is used, namely that for $l_1 \ne l_2$ all the poles of the functions $f_{l_1}(s,g)$ and $f_{l_2}(s,g)$ cannot coincide.

3. Let H denote the Hilbert space of functions square-summable on the interval $(-1,1)$. With the aid of Lemmas 1 and 2 one can prove Theorems 1 and 2.

Theorem 1. The system of functions $f^n(s,z)$, $n = 0, 1, 2 \ldots$, is complete in H.

Let us also define a sequence of functions $\varphi_l(s,z)$ by the formulas

$$\varphi_l(s,z) = \sum_{n=0}^{\infty} \tilde{f}_l^n(s)\sqrt{2(2l+1)}\sqrt{\frac{2n+1}{2}}\,P_n(z),$$

$$\tilde{f}_l^n(s) = \frac{1}{2}\int_{-1}^{1} \tilde{f}^n(s,z)P_l(z)\,dz.$$

Theorem 2. The system of functions $\varphi_l(s,z)$, $l = 0, 1, 2, \ldots$, is complete in H.

Since $H_0 \subset H$, the systems of functions $f^n(s,z)$ and $\varphi_l(s,z)$ are complete in the sense of the metric of H in the space of amplitudes H_0.

Denote by $E_{1/2}$ the ellipse with foci at the points $(-1,0)$, $(1,0)$ and with major semiaxis equal to $\operatorname{ch}((\alpha-\varepsilon)/2)$. Using Hilbert spaces of functions holomorphic inside the ellipse $E_{1/2}$ (^{4}), one can approach the question of the completeness of the system of functions $f^n(s,z)$ in H_0 in the sense of uniform convergence.

Theorem 3. The system of functions $f^n(s,z)$ is complete in H_0 in the sense of uniform convergence inside the ellipse $E_{1/2}$.

For the proof of these theorems, the methods developed by A. N. Markushevich (^{5}) and M. G. Khaplanov (^{6}) are used.

Thus, the proof of Theorems 1 and 2 is carried out according to the following scheme. The operator $A = (a_{ln})$, $a_{ln} = \tilde{f}_l^n\sqrt{2(2l+1)}$, is constructed; it is defined in H and is completely continuous. According to Lemmas 1 and 2, the operators A and A^* have no eigenvalues equal to zero. The functions $\tilde{f}^n(s,z)$ and $\varphi_l(s,z)$ are defined with the aid of the operators A and A^* as follows:

$$\tilde{f}^n = Ae_n, \qquad \varphi_l = A^*e_l, \qquad e_l = \sqrt{\frac{2l+1}{2}}\,P_l,$$

and since the system of functions e_l is complete in H, and the operators A and A^* have no eigenvalues equal to zero, the systems of functions

$$\tilde{f}^n(s,z) = \frac{1}{M_nL_n}f^n(s,z)$$

and $\varphi_l(s,z)$ are complete in H.

Theorem 3 is proved according to the same scheme.

The following question remains open: whether the system of functions $f^n(s,z)$ is a basis in H and in H_0.

4. It seems to us that the results presented in this note on the completeness of perturbation-theory amplitudes in the space of amplitudes may prove useful in determining the scattering amplitude in specific problems. Since

> the scattering amplitude can be represented through linear combinations of the functions $f^n(s,z)$, in order to determine the scattering amplitude it is sufficient to compute the coefficients of $f^n(s,z)$ in the expansion of the scattering amplitude in terms of $f^n(s,z)$.

It is natural to suppose that amplitudes for various processes (with the participation of an arbitrary number of particles) can be expanded in terms of contributions from a definite sequence of Feynman diagrams of those same processes.

Institute of Mathematics
Academy of Sciences of the Ukrainian SSR

Received
27 IV 1966

REFERENCES

1. H. Lehmann, Nuovo Cimento, **10**, 579 (1958).
2. L. Brown, D. I. Fivel et al., Phys., **23**, 187 (1963).
3. B. W. Lee, R. F. Sawyer, Phys. Rev., **127**, 2266 (1962).
4. K. M. Filippov, DAN, **107**, No. 1, 24 (1956); **107**, No. 2, 205 (1956).
5. A. N. Marchukevich, Matem. sborn., **17**, 211 (1945).
6. M. G. Khaplanov, DAN, **80**, 117 (1951), **83**, 35 (1952).

Note: Figure translations are in progress. See original paper for figures.