# Regularization of certain pursuit problem

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## Abstract

The problem of the minimax time $T$ until encounter with respect to a subset of selected coordinates is considered for two linear controllable objects described by identical equations:

$$\begin{gathered}\dot{y} = Ay + Bu \quad \dot{z} = Az + Bv, \\ y_{i_k}(\tau + T^0) = z_{i_k}(\tau + T^0), \quad T^0 = \min_u \max_{v T_{u,v}} \\ (i = 1, \dots, n; k = 1, \dots, m \le n).\end{gathered} \tag{1}$$

It is assumed that the control actions are constrained by integral conditions of the form:

$$\int_\tau^\infty \|u(t)\|^2\,dt \le \mu^2(\tau), \quad \int_\tau^\infty \|v(t)\|^2\,dt \le \nu^2(\tau). \tag{2}$$

The problem (1) under consideration has no solution under conditions (2). The possibility of regularizing this problem is proven. In an effective form, by means of a slight increase in the pursuer' s resource $\mu(\tau)$, an optimal $\mathcal{R}$-strategy is constructed, which ensures a result arbitrarily close to $\inf_u \sup_v T$.

## Full Text

## Preamble

This work continues the investigation of differential games and control problems under uncertainty, building upon the foundations established in [1-6]. We consider the motion of two controlled objects, $y(t)$ and $z(t)$, whose dynamics are governed by the following systems of differential equations:

$$\dot{y} = Ay + Bu \quad (1.1)$$

$$\dot{z} = Cz + Dv \quad (1.2)$$

where $y$ and $z$ are $n$-dimensional state vectors, and $u = \{u_j\}$ and $v = \{v_j\}$ are control vectors for the pursuer and the evader, respectively. The control constraints are defined by the integral norms (see [3], p. 209):

$$\int_{\tau}^{\vartheta} \|u(t)\|^2 dt \leq \mu^2(\tau), \quad \int_{\tau}^{\vartheta} \|v(t)\|^2 dt \leq \nu^2(\tau) \quad (1.3)$$

where $\mu(\tau)$ and $\nu(\tau)$ represent the available resource capacities at time $\tau$. The objective of the pursuer $y$ is to achieve a state $y(\vartheta) = z(\vartheta)$ at some terminal time $\vartheta$, while the evader $z$ attempts to prevent this encounter or maximize the distance at the terminal time.

Let the current state at time $\tau$ be given by $y(\tau)$ and $z(\tau)$. We define the miss distance at the terminal time $\vartheta$ as $x(\vartheta) = y(\vartheta) - z(\vartheta)$. The optimal strategies $u[t]$ and $v[t]$ are sought in the class of feedback controls:

$$u[t] = u[y(t), z(t), \mu(t), \nu(t)], \quad v[t] = v[y(t), z(t), \mu(t), \nu(t)] \quad (1.4)$$

The game is considered over the interval $[\tau, \vartheta]$. Following the methods in [5], we assume the existence of a value function $G(y, z, \mu, \nu, \vartheta)$ that satisfies the corresponding Hamilton-Jacobi-Isaacs equations.

## 1. Optimal Pursuit Strategies

For the linear systems (1.1) and (1.2), the predicted terminal states at time $\vartheta$ can be expressed using the fundamental transition matrices $X(t)$ and $Z(t)$. Let $s = \text{rank } K > m$, where $K$ is the controllability matrix. The pursuer' s strategy is determined by the resource constraint (1.3) and the requirement to minimize the norm of the terminal difference $\|x(\vartheta)\|$. We define the auxiliary function:

$$H(m, \vartheta - t) = X(\vartheta - t)B \quad (1.8)$$

The optimal control $u^0(t)$ that minimizes the energy functional while reaching the target is given by:

$$u^0(t) = H^T(\vartheta - t)D^{-1}(\tau, \vartheta)x(\tau) \quad (1.9)$$

where $D(\tau, \vartheta)$ is the Gramian matrix:

$$D(\tau, \vartheta) = \int_{\tau}^{\vartheta} H(m, \vartheta - t)H^T(m, \vartheta - t)dt \quad (1.10)$$

The condition for successful capture at time $\vartheta_0$ is defined by the equality of the effective resources:

$$\mu^2(\tau) - \nu^2(\tau) = x^T(\tau)D^{-1}(\tau, \vartheta_0)x(\tau) \quad (1.17)$$

where $x(\tau) = y(\tau) - z(\tau)$ represents the current discrepancy in the predicted terminal positions. If such a $\vartheta_0$ exists, the pursuer can guarantee capture regardless of the evader' s actions, provided the evader' s total resource does not exceed $\nu(\tau)$.

### 2. Numerical Example and Simulation

Consider a specific case where the dynamics are defined by second-order integrators:

$$\dot{y}_1 = y_3, \quad \dot{y}_3 = u_1; \quad \dot{y}_2 = y_4, \quad \dot{y}_4 = u_2 \quad (2.1)$$

$$\dot{z}_1 = z_3, \quad \dot{z}_3 = v_1; \quad \dot{z}_2 = z_4, \quad \dot{z}_4 = v_2 \quad (2.2)$$

The initial conditions at $t = 0$ are set as $z_i(0) = 0$, $y_1(0) = y_{10}$, and $y_3(0) = y_{30}$. The evader' s strategy $v(t)$ is assumed to be zero for $t > t^*$. Under these conditions, the optimal pursuit time $T^0$ is found by solving the transcendental equation (2.4):

$$\xi^2 T^3 - 3(x_1 + x_3 T)^2 - 3(x_2 + x_4 T)^2 = 0 \quad (2.4)$$

where $\xi^2 = \mu^2 - \nu^2$. Numerical results for a specific set of parameters (2.11) show that the pursuer successfully reduces the distance to zero at $T^0 = 0.5$. The behavior of the value function and the switching surfaces are illustrated in [FIGURE: 1].

### 3. Stability and Approximation

In practical applications, the exact values of the resources $\mu$ and $\nu$ may be known only with some error $\epsilon(\tau)$. We introduce a modified strategy $u_\epsilon[t]$ that accounts for these perturbations. Let $\delta = \mu - \nu - \epsilon$. We define the regularized control law:

$$u_\epsilon[t] = R[\epsilon, \xi] u^0[t] \quad (4.11)$$

where $R[\epsilon, \xi]$ is a smoothing operator that prevents singularities as the resources are depleted. As shown in (4.19), when $\epsilon \to 0$, the regularized strategy converges to the optimal strategy $u^0$. The stability of this approach is verified by constructing a Lyapunov-like function $V(\epsilon, \xi)$ and demonstrating that $dV/dt < 0$ along the trajectories of the system.

### 4. Conclusion

The proposed feedback control laws (1.19) and (1.21) provide an effective mechanism for pursuit in linear differential games with integral constraints. The inclusion of a regularization parameter $\epsilon$ ensures the robustness of the control system against measurement noise and estimation errors in the resource levels. Further research will focus on extending these results to systems with non-linear dynamics and state constraints.

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## Figures

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# REGULARIZATION OF ONE PURSUIT PROBLEM

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In this article, the features of the pursuit problem [1—6] for same-type objects are discussed in the case where the goal of pursuit is an encounter not in all [3], but only in part of the phase coordinates. A proof of the results announced in the note [4] is given.

§ 1. Let us assume that the change in the phase vectors $\mathbf{y}(t) = \{y_i(t)\}$ and and $\mathbf{z}(t) = \{z_i(t)\}$ $(i = 1, \ldots, n)$ — respectively of the pursuer and the pursued objects — is determined in time by systems of linear differential equations

$$\mathbf{y}' = Ay + Bu, \tag{1.1}$$

$$\mathbf{z}' = Az + Bv, \tag{1.2}$$

where $\mathbf{u} = \{u_j\}$ and $\mathbf{v} = \{v_j\}$ $(j = 1, \ldots, r \leqslant n)$ are control vectors, constrained by integral conditions of the form (see [3], p. 209)

$$\int_{\tau}^{\infty} \|\mathbf{u}(t)\|^2 dt \leqslant \mu^2(\tau), \quad \int_{\tau}^{\infty} \|\mathbf{v}(t)\|^2 dt \leqslant \nu^2(\tau). \tag{1.3}$$

$A$ and $B$ are constant matrices of corresponding dimensions.

Let the selected phase coordinates $y_{ik}$ and $z_{ik}$ $(k = 1, \ldots, m \leqslant n)$, $\ldots$, $n = n$, whose coincidence at the moment of encounter $t = \theta$ constitutes the goal of pursuit, be distinguished. Without loss of generality, we can assume, that selected coordinates are the first $m$ coordinates of the phase vectors $\mathbf{y}$ and $\mathbf{z}$. It is convenient to consider the sets of coordinates $\{y_i\} = \mathbf{y}_{[m]}$ and $\{z_i\} = \mathbf{z}_{[m]}$ $(i = 1, ., m)$ as vectors $\mathbf{q} = \{q_i\}$ $(i = 1, \ldots, m)$ in the $m$-dimensional space $Q$.

Let us approach the problem of realizing an encounter of opposing objects as a differential positional game of two persons with full information [1—6]. In such a game, at each given moment of time $t = \tau$, both partners know all the phase coordinates $y_i(\tau)$, $z_i(\tau)$ $(i = 1, \ldots, n)$, as well as estimate $\mu(\tau)$ and $\nu(\tau)$ remaining for for the time $t \geqslant \tau$ of control resources (1.3). However, information about the present and future choice of $v(t)$ $(t \geqslant \tau)$ is absent. Moreover, the payoff of the game is the time $T_{u,v} = \theta_{u,v} - \tau$ until the encounter of motions $\mathbf{y}(t)$ (1.1) and $z(t)$ (1.2) in part of the selected coordinates. Consequently, the first player (pursuuer) strives to minimize, and the sterond player (pursued) strives to maximize the specified quality index of the game. By the nature of a positional game, the control $\mathbf{u}(\tau)$ at each moment of time $t = \tau$ must most naturally be formed according to the principle of feedback based on the measurement of quantities $y(\tau)$, $z(\tau)$, $\mu(\tau)$ and $\nu(\tau)$, i.e.

Figure 1: Figure 1

functions. At the same time, the study of the stability of the system was reduced to clarifying the continuity of the mapping of the space of generalized processes given by it, equipped with one or another type of convergence. On this path, in note [4], the concept of stability with respect to generalized perturbations was introduced. However, having devoted a large part of the attention to the mathematical side of the problem, the author omitted the presentation of those views on classical stability that served as a starting point in determining the types of stable transformations of generalized perturbations carried out by dynamic systems. The purpose of this note is to fill the gap that has formed, i.e., to clarify the classical content of the concept of stability with respect to generalized perturbations introduced in [4].

### § 1. DISCUSSION OF CLASSICALLY STABLE SYSTEMS FROM THE POINT OF VIEW OF THE TOPOLOGICAL CHARACTER OF THE MAPPINGS OF THE SPACE OF FUNCTIONS OF LOCALLY FINITE VARIATION GENERATED BY THEM *)

Let a system of differential equations be given

$$\frac{d\mu}{dt} = A(t)\mu, \tag{1.1}$$

where $\mu$ — $n$-dimensional vector; $A(t)$ — $n\times n$-matrix with variable elements. We will assume that the matrix $A(t)$ is such that the solution to the Cauchy problem is possible in the entire Euclidean space $E_n$ and is extendable for $t \geq 0$. Then the solution with the initial state $\mu_0$ can be represented in the form $\mu = U(t)\mu_0$, where $U(t)$ denotes the fundamental solution matrix of the system (1.1). Along with the system (1.1), we will consider the perturbed system, the equation of which we write in the form of the following differential relation:

$$d\mu = A(t)\,dt\,\mu + d\eta. \tag{1.2}$$

In the system (1.2), the function $\eta$ describes the perturbing action. Let us assume that it belongs to the space $V(\infty)$. Then the solution to the Cauchy problem exists and is given by the Cauchy formula, which, using the concept of the Stieltjes integral, we write in the form

$$\mu(t) = W(t, 0)\mu_0 + \int_0^t W(t, s)\,d\eta(s). \tag{1.3}$$

In relation (1.3) $W(t, s) = U(t)\,U^{-1}(s)$ — the Cauchy operator.

In the problem of stability with respect to constantly acting perturbations, perturbations possessing sufficiently good differential properties (for example, absolutely differentiable) are considered, and it is assumed that at the initial moment of time the system is at the beginning of coordinates, i.e., $\mu_0 = 0$.

Let us turn to Lyapunov stability [3]. In this case, we have $\eta(t) \equiv 0$ for $0 < t < \infty$ and in the formula (1.3) the integral term vanishes. However, its first term allows the expression [1]

$$W(t, 0)\mu_0 = \int_0^t W(t, s)\,d\chi(s)\mu_0, \quad t > 0,$$

*) As in works [4, 5], in this note we will denote such a space by $V(\infty)$; at the same time if $\eta \in V(\infty)$, then $\eta(t) = 0$ for $t \leqslant 0$.

Figure 2: Figure 2

for system (1.1) when $u = 0$; $l$ — $m$-dimensional vector, the sign ' denotes transposition. Then, by the Cauchy formula, we obtain

$$y[m](\theta) = X[m](\theta - \tau)\, y(\tau) + \int_{\tau}^{\theta} H[m](\theta - t)\, H[m]'(\theta - t)\, l\, dt, \quad (1.9)$$

there $X[m]$ — matrix, composed of the first $m$ rows of the fundamental matrix $X$. From (1.9) we can find

$$l = D^{-1}\varphi[m]. \quad (1.10)$$

Here $D^{-1}$ — matrix, inverse to the matrix

$$D = \int_{\tau}^{\theta} H[m](\theta - t)\, H[m]'(\theta - t)\, dt, \quad (1.11)$$

$$\varphi[m] = y[m](\theta) - X[m](\theta - \tau) \cdot y(\tau), \quad (1.12)$$

and the matrix $D^{-1}$ necessarily exists, if only the system (1.1) is completely controllable in terms of part of the selected coordinates. Considering further (1.7), (1.8), (1.10), we obtain, finally, the relationship defining the region $G^{(1)}[y(\tau), \mu(\tau), \theta]$ in the space $Q$

$$G_1(q_1, \ldots, q_m) \equiv (D^{-1}\varphi[m], \varphi[m]) \le \mu^2. \quad (1.13)$$

In a similar way, one can construct the region $G^{(2)}[z(\tau), \nu(\tau), \theta]$. In the end, we obtain

$$G_2(q_1, \ldots, q_m) \equiv (D^{-1}\psi[m], \psi[m]) \le \nu^2, \quad (1.14)$$

where $\psi[m] = z[m](\theta) - X[m](\theta - \tau)\, z(\tau)$.

Let now $\theta = \theta_0$ — the moment of absorption of the process $z(t)$ by the process $y(t)$, i.e., such a moment $t = \theta$, when for the first time the region $G^{(2)}[z(\tau), \nu(\tau), \theta]$ falls entirely inside the region $G^{(1)}[y(\tau), \mu$ $\theta]$. Obviously, that with continuous change of $\theta$ the bounded closed

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Let now $\theta = \theta_0$ — the moment of absorption of the process $z(t)$ by the process $y(t)$, i.e., such a moment $t = \theta$, when for the first time the region $G^{(2)}[z(\tau), \nu(\tau), \theta]$ falls entirely inside the region $G^{(1)}[y(\tau), \mu(\tau), \theta]$. Obviously, that with continuous change of $\theta$ the bounded
closed and convex regions $G^{(1)}$ and $G^{(2)}$ deform continuously any and therefore it turns out, that at the moment $\theta = \theta_0$ their boundaries touch at least in one point. The functions $G_1$ (1.13) and $G_2$ (1.14) emerged as similar quadratic functions of variables $q_1, \ldots, q_m$, and, consequently, the relatis (1.13) and (1.14) will define in the space $Q$ similar and identically oriented ellipsoids. From here directly follows the following circumstance: if the region $G^{(2)}$ is entirely contained in the region $G^{(1)}$ and their boundaries touch, then this tangency can take place either in a single point $q^0$, or the points of tangency will be infinitely many and then the indicated regions simply coincide.

Let us assume, that such values $y(\tau)$, $z(\tau)$, $\mu(\tau)$, $\nu(\tau)$ were realized, for which the number $\theta_0$ exists and the region $G^{(2)}[z(\tau), \nu(\tau), \theta_0]$ touches the region $G^{(1)}[y(\tau), \mu(\tau), \theta_0]$ in a single point $q^0$. To determine this point, it is sufficient [5] to solve the following problent for conditional minimum:

$$\min_{[\varrho]} G_2(q_1, \ldots, q_m) = \nu^2 \quad (1.15)$$

under $G_1(q_1, \ldots, q_m) = \mu^2$. Solving the problem (1.15) by the usual method of Lagrange multipliers, we find

$$q^0 = \frac{1}{\mu - \nu} X[m](\theta_0 - \tau)\, [\mu z - \nu y]. \quad (1.16)$$

Figure 3: Figure 3

Simultaneously, we obtain a finite equation for determining the moment of absorption

$$(D^{-1} C_{[m]}, C_{[m]}) - (\mu - \nu)^2 = 0, \tag{1.17}$$

where

$$C_{[m]} = - X^{[m]} (\vartheta - \tau)\, x(\tau),\ x(\tau) = y(\tau) - z(\tau). \tag{1.18}$$

The moment of absorption will be the *smallest* positive root $\vartheta = \vartheta_0$ of equation (1.17). Let us now construct the controls $u_0[t]$ and $v_0[t]$, aiming (see [5], p. 7) the movements $y(t)$ and $z(t)$ at each moment of time $t = \tau$ to the point $y_{[m]}(\vartheta_0) = z_{[m]}(\vartheta_0) = q^0$ (1.16). Subsequently, these controls are called *extremal*, and the rule for aiming movements to the point $q^0$ is the *rule of extremal aiming*. Taking into account (1.8), (1.10), (1.12), (1.16), we obtain

$$u_0[t] = u_0 [y(t), z(t), \mu(t), \nu(t)] = \frac{\mu}{\mu - \nu} w^0[t], \tag{1.19}$$

where

$$w^0[t] = H^{[m]} D^{-1} C_{[m]}. \tag{1.20}$$

Similarly, we find

$$v_0[t] = v_0 [y(t), z(t), \mu(t), \nu(t)] = \frac{\nu}{\mu - \nu} w^0[t]. \tag{1.21}$$

By direct calculations, it can be established that $w^0[t]^0$ (1.20) is a solution to the problem of transferring the system

$$\dot{x} = Ax + Bw \tag{1.22}$$

from state $x = x(\tau)$ to position $y_{[m]}(\vartheta^0) - z_{[m]}(\vartheta^0) = x_{[m]}(\vartheta^0) = 0$ under the restriction

$$\left[\int^{\infty} [w(t)]^2\, dt\right]^{1/2} \leqslant \zeta(\tau) = \mu(\tau) - \nu(\tau) \tag{1.23}$$

and under the condition

$$T^0 = \vartheta^0 - \tau = \min_{w} T. \tag{1.24}$$

In particular, it also follows that the moment of absorption $\vartheta_0$ determined from (1.17) cobnades with the moment $\vartheta^0$ of the arrival of the movement $x(t)$ to the position $x_{[m]} = 0$.

In paper [3], where it was a question of meeting with respect to all phase coordinates ($m = n$), it was established that the extremal controls $u_0$ (1.19) and $v_0$ (1.21) are optimal strategies, that solve problem 1 on pursuit. This fact took place because with $u = u_0$ (1.19) and for anny admissible $v$, $v$, during the time before the meeting, a situation could not arise when the brondaries of the attainability sets $G^{(1)}$ (1.13) and $G^{(2)}$ (1.14) touch at more than one points, unless only at the initiarial moment of pursuit $t = t_0$ the specified contact occurred at a single point $q^0 q^0(t_0)$ (1.16). The sittation is more complicated in the considered cavae $m < n$. As follows from the subsequent, here the extremal control $u_0$ (1.19) no longer guarantees for any any admyssible control $v$ a meeting of movements $y(t)$ and $z(t)$ during the time $t \leqslant t \leqslant \tau + T^0(\tau)$, and the rule of extremal aiming does not encsture $\min_u \max_v T = T^0$. In other words, the extremal controls $u = u_0$ (1.19) and $v = v_0$ (1.21) do not constieute a pair of optimal strategies for $m < n$. This statement is proved by the following example.

Figure 4: Figure 4

§ **2.** Let systems (1.1) and (1.2) have the form

$$\dot{y}_1 = y_3,\ \dot{u}_3 = u_1,\ \dot{y}_2 = y_1,\ \dot{y}_1 = u_2, \tag{2.1}$$

$$\dot{z}_1 = z_3,\ \dot{z}_3 = v_1,\ \dot{z}_2 = z_1,\ \dot{z}_1 = v_2. \tag{2.2}$$

and it is required to bring about a meeting only along coordinates $y_1$, $y_2$ and $z_1$, $z_2$. The extremal control $u_0$ as a consequence of (1.19), (1.20) has the form

$$u_0 = \left\{ -\frac{3}{|T^\circ|^2}\ \frac{\mu}{\zeta}\,(x_1 + T^\circ x_3);\ -\frac{3}{|T^\circ|^2}\ \frac{\mu}{\zeta}\,(x_2 + T^\circ x_1) \right\}, \tag{2.3}$$

where $\zeta = \mu - \nu$, and the quantity $T^\circ$ is the smallest positive root of the equation (1.17)

$$\zeta^2 T^\circ - 3\,(x_1 + x_3 T)^2 - 3\,(x_2 + x_4 T)^2 = 0. \tag{2.4}$$

Let us assume that at the initial moment of pursuit $t = t_0 = 0$ the position took place

$$z_1(0) = z_2(0) = z_3(0) = z_1(0) = 0,$$
$$y_2(0) = y_4(0) = 0,\ y_4(0) = y_{10},\ y_3(0) = y_{30}. \tag{2.5}$$

and, in addition, let us assume that the evader chose for some time $t_0 < t < t^* < \vartheta$ control $\nu(t) = \{v_1(t),\ v_2(t)\} \equiv 0$. Then for all time, while $\nu(t) \equiv 0$, the equalities will be fulfilled $z_1(t) = z_2(t) = z_2(t) = z_4(t) = z_4(t) = y_2(t) = y_4(t) = 0$, $\nu(t) = \nu(t^*) = \nu_0$, and the process of pursuit, conducted by the pursuer according to the rule of extremal aiming (2.3), will be described by the system of differential equations

$$\begin{aligned} \dot{y}_1 &= y_3, \\ \dot{y}_3 &= -\sqrt{3}\ \frac{\mu}{\sqrt{T^\circ}}\ \operatorname{sgn}(y_1 + y_3 T^\circ), \\ \dot{\mu} &= -\frac{3\mu}{2T^\circ}, \\ \mathring{T}^\circ &= -1 - \frac{\nu_0}{(\mu - \nu_0) - \dfrac{2}{\sqrt{3}}\, y_0\, \dfrac{1}{\sqrt{T^\circ}}\ \operatorname{sgn}(y_1 + y_3 T^\circ)}. \end{aligned} \tag{2.6}$$

The last differential equation in system (2.6) is obtained by formal calculation of the derivative $dT^\circ/dt$ implicitly from equation (2.4). Let us now try to indicate such initial conditions

$$y_1(0) = y_{10},\ y_a(0) = y_{30},\ \mu(0) = \nu_0 > \nu_0,\ T^\circ(0) = T_0, \tag{2.7}$$

for which at the moment of time $t = t^*$ by virtue of differential equations (2.6) one gets $\mu(t^*) = \nu(t^*) = \nu_0$. In such a case the attainability sets $G^{(1)}[y(t^*),\ \mu(t^*),\ \vartheta_0(t^*)]$ (1.13) and $G^{(2)}[z(t^*),\ \nu(t^*),\ \vartheta_0(t^*)]$ (1.14), being for our example circles of radii $\{[T^\circ]^3\mu^2/3\}$ and $\{[T^\circ]^3\nu^2/3\}$ respectively, will turn out to be coinciding.

Let us assume that the required initial conditions (2.7) exist and the moment $t = t^*$ has arrived, when $\mu(t^*) = \nu(t^*)$. Then from (2.4) will follow the equality

$$\lambda(t_*) = y_1(t^*) + y_3(t_*)\,T^\circ(t_*) = 0. \tag{2.8}$$

Figure 5: Figure 5

inkaisating the coindance of the centers of the cirycss indisated above. Lect $v_0 = v(t^*) = 1$, $y_1(t^*) = -0.25$, $T^0(t^*) = 0.25$, we get from (2.8) $y_3(t^*) = 1$. Tnep lyte's take the values

$$y_1(t^*) = -0.25,\ y_3(t^*) = 1,\ \mu(t^*) = 1,\ T^0(t^*) = 0.25 \qquad (2.9)$$

as new initial conditions and will, proceeding from them, integrate system (2.6) basadings, setting, cendosarnently, $t = -t' + t^*$. If it in this clyvae ot turns out, that for the initalial dans (2.9) under obpatse solution of the sustem (2.6) cymectvess, torco, onevisinlo, as the uckomed toan initalizal dans (2.7) onn mown trirate mo6se shavenes of the incuerent peluetion for $t' > t'_0 = t^*$. It is unllind to yctanoble, that the pelution of inbertoyed sustems (2.6) for the invalisal yclobuins (2.9) cymectvess and is in edunceronamic at lense sor $t' > t'_0$, doctatowly briske ko $t'_0$, echu only one redetermines sign of beluvinng $\lambda(t')$ for $t = t^*$, secuning sgn $\lambda(t^*) = -1$. It octaics to npocerte, then, the beluvinrg $T^0(t')$ for $t' > t'_0$, onderelined the uncleshed equation of custem (2.6), is nwt the naimenilen polaritative ropn of the ypabneting

$$F(t', T) = \xi^2(t')T^3 - 3\,[y_1(t') + y_3(t')T]^2 = 0. \qquad (2.10)$$

This is nigthermised by a direct numerical excepcinement. Eclra custem (2.6) integropsed in obpatse speme with the initialized dannais (2.9), havining from moment $t'_0 = 0$ u, for example, up to moment $t'_s = 0.16$, tora a pesult of the calculation it is nolyven

$$y_1(t'_s) = -0.358510, \quad y_3(t'_s) = 0.338838,$$

$$\mu(t'_s) = 1.946702, \qquad T^0(t'_s) = 0.500000. \qquad (2.11)$$

At the same time in oberns out, that the nolyvened soliening $T^0(t')$ at karigh moment of spme $0 < t' < 0.16$ is the maimenishest poloritverative ropn of equation (2.10). On fig. 1 is presctent the deformation of curvail $F = F(t', T)$ with the change of $t'$ on the etgreskt [fig. 1].

Thus, it have the uckowed initializeal dannais (2.7) one can write, for example, invalues (2.11)

$$y_{10} = y_1(t'_s),\ y_{30} = y_3(t'_s),\ \mu_0 = \mu(t'_s),\ T_0 = T^0(t'_s).$$

Intergiping the system (2.6) with these navalimsl yclobuiams, at moment $t = t'_s = 0.16$ we holyvein, them, thus, $\mu(t^*)\ v(t^*)$, and, cendosatenly, oblects of reachabmosts $G^{(1)}$ and $G^{(2)}$ will obern oxat at this moment of spme enarged. But at such moments of spme, when the oblects of reachabimosts tocument But at such a point, roxin, rule extremahism of armortraming ctonmated, obviously, strategy and ( Morove, as takix moments of spme it troes impossible to judicate stpaterogy $u$ (1.4), which would keepar the oblects $G^{(2)}$ inside the oblectn $G^{(1)}$, eccin, if only the pursuer is not informed about the esloice by the pursued at these moments of spene the control $v(r\imath)$. Consequently, extremalizant control $u_0$ (1.19) for $m < n$, ogherly robopir, does not garantize the sctpting drations at the moment of time $t = \vartheta_0$.

§ 3. Let us new apsume, that there exists an admissible control of the form $u(\tau) = u^*\,[y(\tau), z(\tau), \mu(\tau), v(\tau)]$, which obscoreves the sctpting of drations by parts of choinen coordinates at the moment $t < \tau + T^0(\tau)$ with any adnyssinirable behavioror of the pursuies. At the same time in coordertive with the grpe formulation of problem 1 it is necessary to pascmater, the that the pursuer is not garantized from the choice by the pursued of control

Figure 6: Figure 6

$$\nu = \nu^* = \frac{\nu}{\mu} u^*. \tag{3.1}$$

Hot then in the process of drtation for $t \geqslant \tau$, the paremity will always support both

$$\nu(t)/\mu(t) = \nu_0/\mu_0 = \text{const}, \tag{3.2}$$

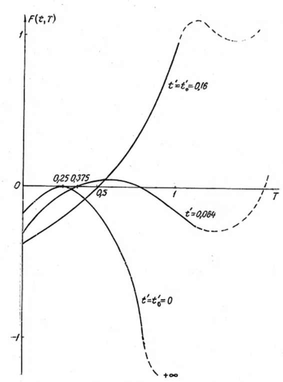


Fig. 1

which is onlyned as a condstence of intergring the ypanions

$$\dot{\mu} = -\|u\|^2/2\mu, \quad \dot{\nu} = -\|\nu\|^2/2\nu, \tag{3.3}$$

derived from the restractivetive corilitions (1.3). From pavestry (3.2) is deryived, that the controler $\nu = \nu^*$ (3.1) is take adnyctiable and, moreve, teo, that on intervale $[\tau, \tau + T^0(\tau)]$, the nepavensty will repavencct

$$\left[\int_{\tau}^{\tau+T^0(\tau)} \|w^*[t]\|^2 dt\right]^{1/2} < \zeta(\tau) = \mu(\tau) - \nu(\tau), \tag{3.4}$$

rde $w^* = u^* - \nu^*$. Иs nepaventry (3.4) it colleys, that an actier is nesosnsible for $t < \tau + T^0(\tau)$, sincornky $T^0 = \min_\omega T$ is the pslution formytoyportated

Figure 7: Figure 7



in § 1 of the problem of limiting time-optimality (see (1.22), (1.23), (1.24)). Meeting could have occurred precisely at the moment $t = \tau + T^0(\tau)$, only if $w^*[t]$ coincided with $w^0[t]$ (1.20), but then the control $u^{*}\,{}^*[t]$ would be extremal, and the extremal control, as follows from the example considered in the previous section, does not ensure meeting at the moment $t < \tau + T^0(\tau)$.

Thus, we come to the conclusion that it is generally impossible to construct a strategy of the form $u(\tau) = u[y(\tau), z(\tau), \mu(\tau), \nu(\tau)]$, which would guarantee the meeting of movements by part of the phase coordinates at the moment $t < \tau + T^0(\tau)$ under any admissible behavior of the pursued. This means that problem 1 on pursuit for $m < n$ turns out to be ill-posed and, consequently, the necessity of its regularization arises.

§ 4. Let us dwell here on one method of regularization of the pursuit problem. In doing so, we will be guided by the considerations s will be guided by the considerations expressed in work [6], (see § 5). Let us assume that the control resource of the pursuer is somewhat increceased. For convenience of notation, we assume that $\mu(\tau)$ is already try the reserve, increased by a small number $\varepsilon(\tau), (\tau) > 0$. We construct the attainability sets $G^{(1)}[y(\tau), \mu(\tau) - \varepsilon(\tau), \theta\epsilon]$ and $G^{(2)}[z(\tau), \nu(\tau), \theta\epsilon]$ (see Fig. 2). Here $\theta_\varepsilon$ is the moment of absorption of the process $z(t)$ by the process $y(t)$ under the condition that the pursuer posseses the Obviously, the set $G^{(2)}[z(\tau), \nu(\tau), \theta\epsilon]$ will be strictly inside the set $G^{(1)}[y(\tau), \mu(\tau), \theta\epsilon]$, touching the boundary of the set $G^{(1)}[y(\tau), \mu(\tau) - \varepsilon(\tau), \theta\epsilon]$ at the point $q_{0\varepsilon}$.

Let us assume that we managed to carry out such a choice of control $u$, for which the value $\varepsilon(t)$ for $t > \tau$ remains positive all the time until the meeting. This will mean that the set $G^{(2)}[z(t), \nu(t), \theta e]$ remains strictly inside the set $G^{(1)}[y(t), \imath(t), \theta e]$ remots the set $G^{(1)}[y(t), \mu(t), \theta\epsilon]$ all the time until the meeting and, consequently, with a non-increeasing absorption moment $\theta_\varepsilon(t)$ the meeting will occur no later than at the moment $t = \vartheta_\varepsilon(\tau)$.

$\mu(\tau) - \xi\varepsilon]$ at the point $q_{0\epsilon}$. Let us assume that we managed to carry choice of control $u$, for which the value value $\varepsilon(t)$ for $t > \tau$ remains positive nowards all the time until the meeting. This will mean that the set $G^{(2)}[z(t), \nu(t), \vartheta_\epsilon]$ remains strictly inside the set $G^{(1)}[y(t), \mu(t), \theta_e]$ all the time until the meeting and, consequently, with a non-increasing consequently, with a non-increasing absorption moment $\theta_e(t)$ the meeting will occur no later than at the moment $t = \theta_e(t)$.

It turns out that such a choice of control $u$ is possible in the form

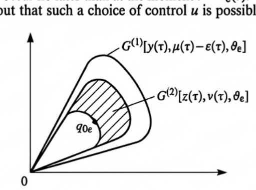


Fig. 2

It turns out that such a choice of control $u$ is possible in the form (tmr exmt meaning of this momewn or conpen at twst the form (The exact meaning of this somewhat conditional entry will be revealed below, where the entry (4.1) is considered as limiting for some discrete scheme). At the same time, the algorithm for calculating the control forces $u[\tau]$ can rely only on the values $y(\tau), z(\tau), \mu(\tau), \nu(\tau)$, realized in the pursuit process. The strategy $u[\tau]$ (4.1) will be called in the further *R-strategy*.

Let the tangency of the boundaries of the sets $G^{(1)}[y(\tau), \mu(\tau) - \varepsilon(\tau), \theta\epsilon]$ and $G^{(2)}[z(\tau), \nu(\tau), \theta\epsilon]$ be carried out at a single point $q_{0e}$, then it is possible to determine the extremal controls, aiming the movements $y(t)$ and $z(t)$ to the point $q_{0e}$.

Equations (4.1) and (4.2) remain unchanged. (4.2)

Figure 8: Figure 8

For the absence of movable critical singular points, it is necessary [1] $(n+1)\delta_1 \leqslant 2\delta_1 - 2$, but this cannot be for $n > 0$.

**Lemma 4.** In order for the system of form (II) not to contain movable critical singular points, it is necessary $\frac{\partial Q_2}{\partial x} \equiv 0$.

Let Lemma 4 hold, i.e.,

$$\frac{dx}{dz} = \sum_{k=0}^{n} p_k(z) y^k, \quad \frac{dy}{dz} = \sum_{k=0}^{N} q_k(z) x^k. \tag{28}$$

If $n = N = 1$, then the system is linear and, therefore, does not contain movable singular points at all.

Let $nN \neq 1$. Introduce the parameter $\lambda$:

$$x = \frac{1}{\lambda^{n+1} T}, \quad y = \frac{1}{\lambda^{n+1} W}, \quad z = z_0 + \lambda^{Nn-1} t,$$

for $\lambda = 0$ we get

$$\frac{dT_0}{dt} = -\frac{T_0^2}{W_0^n} p_n(z_0), \quad \frac{dW_0}{dt} = -\frac{W_0^2}{T_0^N} q_N(z_0). \tag{29}$$

Consider a one-parameter family of solutions of system (29)

$$T_0 = A(t+C)^{\frac{n+1}{Nn-1}}, \quad W_0 = B(t+C)^{\frac{N+1}{Nn-1}}, \tag{30}$$

where $AB \neq 0$ — are constants; $C$ — is an arbitrary parameter. For the uniqueness of $T_0$ and $W_0$, it is necessary to require that $\frac{n+1}{Nn-1} = s$, $\frac{N+1}{Nn-1} = l$, where $s$ and $l$ — are integers, obviously $s > 0$, $l > 0$ and $\frac{1}{s} + \frac{1}{l} + \frac{1}{sl} = Nn - 1$. It is easy to see that the latter is possible only for the following values of $n$ and $N$:

1. $n = 2$, $N = 1$ $(n = 1, N = 2)$.
2. $n = 3$, $N = 1$ $(n = 1, N = 3)$.
3. $n = N = 2$.

Let us have $n = 2$, $N = 1$, i.e.,

$$\frac{dx}{dz} = p_2(z) y^2 + p_1(z) y + p_0(z), \quad \frac{dy}{dz} = q_1(z) x + q_0(z), \tag{31}$$

$$p_2(z) \neq 0, \quad q_1(x) \neq 0.$$

This system can be reduced by a linear transformation to the form

$$\frac{dx}{dz} = \widetilde{p}_2(z) y^2 + \widetilde{p}_1(z) y, \quad \frac{dy}{dz} = x, \tag{32}$$

obviously, (32) is equivalent to the second-order equation

$$\frac{d^2 y}{dz^2} = \widetilde{p}_2(z) y^2 + \widetilde{p}_1(z) y. \tag{33}$$

Figure 9: Figure 9

Let now $0 < \zeta < \varepsilon^2$. Consider on the plane $\{\varepsilon, \zeta\}$ the family of curves defined by the differential equation

$$\frac{d\varepsilon}{d\zeta} = \frac{V\sqrt{\zeta - \varepsilon}}{\varepsilon} \quad (\varepsilon = \bar{\varepsilon} \quad \text{at} \quad \zeta = 0). \tag{4.13}$$

The integral curve passing through ($\varepsilon = f(\zeta)$) and satisfying $\bar{\varepsilon} = f(0) > 0$ is depicted in Fig. 3. However, multiply in the neighborhood, let's integrate over the circle of functions $V(\varepsilon, \zeta) = \varepsilon - f(\zeta)$ by virtue of the differential equations (4.9), (4.10). We will have

$$\frac{dV}{dt} = \frac{\|\mathbf{u}^0\|^2}{2\varepsilon\mu\zeta}\,[\mu(1 + V\overline{\zeta}) - $$
$$- \varepsilon(1 + \mu)] + \frac{1}{2(\nu + \zeta)}\|\delta\mathbf{v} - \delta\alpha\|^2 +$$
$$+ \frac{\xi}{2\nu(\nu + \zeta)}\|\delta\alpha\|^2.$$

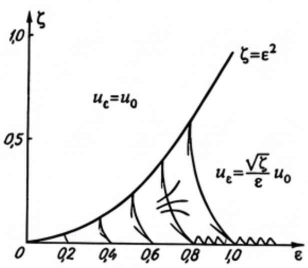

Fig. 3

The quantity $[\mu(1 + V\overline{\zeta}) - \varepsilon(1 + \mu)]$ is positive at least for $\varepsilon$ sufficiently small compared to $\mu$. Consequenty for $0 < \zeta < \varepsilon^2$, the consetuently, for $0 < \zeta < \varepsilon^2$, the derivative $dV/dt$ is non-negative for any $\delta$ and $\mu$, meaning the trajectories of the system of differential equations (4.9), (4.10) cannot cross the aforementioned curves $\varepsilon = f(\zeta)$ for small values of $\varepsilon$ in the direction of decreasing $\varepsilon$, i.e., $\varepsilon(t)$ for $t > t_0$ will remain non-negative, provided that at the initial moment of pursuit $t = t_0$, $\varepsilon(t_0) > 0$.

It remains to consider the case $\zeta = 0$. The construction of the $A$-strategy (4.11), (4.12) function $u_\varepsilon$ is continuous in $\zeta > 0$ and at $\zeta = 0$, $u_\varepsilon = 0$. Moreover, at $\zeta = 0$ we have $\sigma_{\text{tant}} = 0$, then

$$\xi = \lim_{\Delta t \to +0} \frac{\xi - 0}{\Delta t} = \sqrt{(D^{-1}H^{[m]}\mathbf{v}, H^{[m]}\mathbf{v})} \tag{4.14}$$

and in accordance with (4.6)

$$\varepsilon = \|\mathbf{v}\|^2/2\nu - \xi. \tag{4.15}$$

Calculating now, taking into account (4.14), (4.15), the total time derivative of the aforementioned function $V(\varepsilon, \zeta)$, we find that at $\zeta = 0$

$$\frac{dV}{dt} = -\frac{d\varepsilon}{d\zeta}\,\xi + \varepsilon = \xi + \varepsilon = \frac{\|\mathbf{v}\|^2}{2\nu} \geqslant 0$$

in, consequently, at $\zeta = 0$ it changes sign to the opposite.

Thus, each pursuer is guided by the indicated $A$-strategy (4.11), (4.12), then the meeting will necessarily take place regardless of the actions of the evader no later than at moment $\theta_\varepsilon(f_0)$, which serves as the smallest positive root of the equation (4.4) at $\tau = f_0$. Until now, it was assumed that the moment of capture $\theta_s(t)$ remains unchanged during the pursuit process and equal to $\theta_\varepsilon(f_0)$. However, it is necessary to consider it varying, since the evader may behave unreasonably ($\Delta\nu \neq 0$), and then there is a possibility to ensure a meeting strictly before moment $\theta_\varepsilon(f_0)$. In the condi-

Figure 10: Figure 10

conditions of the introduced regularization, the continuous correction of the absorption moment $\vartheta_e(t)$ in accordance with equation (4.4) must be strictly understood as the limiting case of the change in the values of $\vartheta_e$ at discrete moments of time $t_k$ ($k = 1, 2, \ldots$), between which, on small intervals $[t_k, t_{k+1} = t_k + \Delta t)$, the control $u_e[t]$ is determined with a constant $\vartheta_e(t) = \vartheta_e(t_k)$, which is the smallest positive root of the equation

$$x[x(t_k), t_k, \vartheta] - \mu(t_k) + \nu(t_k) + \varepsilon^*(t_k) = 0, \tag{4.16}$$

where

$$\varepsilon^*(t_k) = \min\{\varepsilon(t_0); \varepsilon(t_k) = \mu(t_k) - \nu(t_k) - x[x(t_k), t_k, \vartheta_e(t_{k-1})]\}.$$

Using the property of continuity of the function $x = x[x(\tau), \tau, \vartheta]$ (4.5) with respect to the variable $\vartheta > \tau$ for any fixed $\tau$, $x(\tau)$ and taking into account the limiting relation $\lim x[x(t_k), t, \vartheta] = +\infty$ as $\vartheta \to \tau$, it can be established that $\vartheta_e(t_{k-1}) \geq \vartheta_e(t_k)$ ($k = 1, 2\ldots$) always holds, i.e., the absorption moment, corrected at moments $\tau = t_k$ by equation (4.16), changes over time without increasing. With all this, it is important to note that the value $\varepsilon(t)$ must be set only at the initial moment of pursuit $t = t_0$. Subsequently, it is determined from the relation $\varepsilon(t) = \mu(t) - \nu(t) - x[x(t), t, \vartheta_e]$ based on the measurement of the values $y(t)$, $z(t)$, $\mu(t)$, $\nu(t)$. Now, in accordance with (4.16), in formula (4.12) at moments of time $t = t_k$, it is necessary to assume $\varepsilon = \varepsilon^*(t_k)$.

We also note that with $\zeta = 0$, the appearance of a peculiar sliding regime is possible. Indeed, with $\zeta = 0$, the derivative $\dot{\zeta}$ (4.14) is strictly positive, if only $\nu \neq 0$. Therefore, over a small time interval $\Delta t$, the representative point $(e, \zeta)$ from the axis $\zeta = 0$ gets into the region $0 < \zeta < \varepsilon^2$, where in accordance with (4.9)

$$\dot{\zeta} = (w^0/\zeta, \nu) + o_1(\zeta), \quad o_1(\zeta) \to 0 \text{ as } \zeta \to 0 \tag{4.17}$$

and, consequently, with $\zeta > 0$, the pursued can choose the control $\nu$ (for example, $\nu = -\nu_0$), such that $\dot{\zeta}$ will be negative until $\zeta$ becomes zero again. Moreover, considering (4.10), (4.17), with $\zeta > 0$ we have

$$\frac{d\zeta}{de} = [x(t_k), t_k, \vartheta] - \mu(t_k) + \nu(t_k)\,) - \nu(t_k) - x\left[[x(t), t_k, \vartheta_E] + \frac{d\zeta}{de}, \frac{t/\nu}{\tau}\right] + o_2(\zeta), \tag{4.18}$$

$$o_2(\zeta) \to 0 \text{ as } \zeta \to 0$$

and, taking into account (4.14), (4.15), with $\zeta = 0$ we obtain

$$\frac{d\zeta}{de} = \frac{\dot{\zeta}}{-\dot{\zeta} + \|\nu\|^2/2\nu}. \tag{4.19}$$

From (4.18) and (4.19) it follows that for any $\nu$, for which $(w^0/\zeta, \nu) \leqslant 0$, and as $\zeta \to 0$, the inequality $0 > \frac{d\zeta}{de} \geqslant -1$ holds, and with $\zeta = 0$ the tangent to the curve $\zeta = \zeta(\zeta)$ cannot make with the axis $\zeta = 0$ an angle greater than

Figure 11: Figure 11

than ε 135°. Thus, the possible sliding mode, to which the schematically depicted sawtooth curve corresponds in Fig. 3. Though guards only to an yoeliance in *e*(*f*).

Soo, as a buisod, it can be being assinated, that the control $u_\varepsilon$ (4.11), (4.12) provides the maximum of what the pursuiter may *w* decorate at the view of an arbitrarily small yoelinence in the pursuit resource $\mu$, regular-rizling pursuit applying coinciding the initial coincidence selected coordinates at a moment active $\delta_\varepsilon$, which is a regularizing control insignificantly entirely of moments prolongation $\theta_0$.

However, as indicated by numerical experiments, constructed *R*-strategy (4.11), (4.12), generally general, is new found for practical realization of pursuit processes on computational specifics. More convenient from the tank of set outset to be choice the regularizing function $R[\varepsilon, \zeta]$ in bude

$$R[\varepsilon, \zeta] = \begin{cases} 1 & \text{for } \zeta \geq \varepsilon, \\ \zeta/\varepsilon & \text{for } 0 \leq \zeta < \varepsilon. \end{cases} \qquad (4.20)$$

Let us establish, though in the control using the pursuing *R*-strategy (4.11), (4.20) by pursuiter for choice $\delta v$, function *e*(*f*), which ensures, that can turn a null in final scheme and therefore will be obtained belong positive to semiaxis special. Consider hence consider homogeneous differential equation

$$\frac{d\varepsilon}{d\zeta} = \frac{\zeta - \varepsilon}{\varepsilon} \quad (\varepsilon = \varepsilon^- \text{ for } \zeta = 0). \qquad (4.21)$$

The central integral curves of this equation have the bud

$$(\varepsilon^2 + \varepsilon\xi - \zeta^2)^{1/2} \left(\frac{\varepsilon + a\zeta}{\varepsilon + b\zeta}\right)^{\frac{1}{2(a-b)}} = \varepsilon^-, \qquad (4.22)$$

$$a = \frac{1+\sqrt{5}}{2}, \quad b = \frac{1-\sqrt{5}}{2}.$$

Let us analyse in the object $0 < \zeta < \varepsilon$ null scheme district the element of function $V(\varepsilon, \zeta)$, for which integral curves $\varepsilon = f(\zeta)$ equation (4.21), corresponding to constant values of $\varepsilon^-$, arrives at curves of constant level $V(\varepsilon, \zeta) = \varepsilon^-$. In accordance with (4.22)

$$V(\varepsilon, \zeta) = (\varepsilon^2 + \varepsilon\xi - \zeta^2)^{1/2} \left(\frac{\varepsilon + a\zeta}{\varepsilon + b\zeta}\right)^{\frac{1}{2(a-b)}}. \qquad (4.23)$$

Considering (4.9, 4.10) and (4.20), obtain

$$\frac{dV}{dt} = \frac{\varepsilon V}{\varepsilon^3 + \varepsilon\xi - \zeta^2}\left(\varepsilon - \frac{\zeta - \varepsilon}{\varepsilon}\zeta\right) =$$

$$= \frac{\varepsilon V}{\varepsilon^2 + \varepsilon\xi - \zeta^2}\left[\frac{\|w^0\|^2}{2\zeta^2}\left[\frac{\zeta^2(\mu - \varepsilon)}{\varepsilon\mu} + \frac{(\zeta - \varepsilon)^2(\mu - \varepsilon)}{\varepsilon^2}\right] - \right.$$

$$\left. - \frac{(\varepsilon - \zeta)\zeta}{\varepsilon}\right] + \frac{\|\delta v\|^2}{2v} - \frac{(\zeta - \varepsilon)}{\varepsilon\zeta}(w^0, \delta_v)\Big). \qquad (4.24)$$

Figure 12: Figure 12

Let us estimate in the region $0 < \zeta \leqslant \varepsilon$ the derivative $dV/dt$ (4.24) from below. Taking into account that $\|w^0\|^2/\zeta^2 = k$ is a bounded quantity, and considering that

$$\min_{\delta v} \left[ \frac{\|\delta v\|^2}{2v} - \frac{(\zeta - \varepsilon)}{\varepsilon\zeta} (w^0, \delta v) \right] = - \frac{v(\zeta - \varepsilon)^2}{2\varepsilon^2\zeta^2} \|w^0\|^2,$$

$$\frac{5}{4} \varepsilon^2 > (\varepsilon^3 + \varepsilon\xi - \xi^2) \geqslant \varepsilon^2 \quad \text{npu } 0 < \zeta \leqslant \varepsilon,$$

we find for $\zeta \leqslant \varepsilon \leqslant \mu$

$$\frac{dV}{dt} \geqslant - \frac{k}{2}, V,$$

but in the region $0 < \zeta \leqslant \varepsilon$ we have

$$\varepsilon \left( \frac{3 + \sqrt{5}}{3 - \sqrt{5}} \right)^{\frac{1}{2\sqrt{5}}} \geqslant V(\varepsilon, \zeta) > \varepsilon,$$

consequently,

$$\varepsilon(t) \left( \frac{3 + \sqrt{5}}{3 - \sqrt{5}} \right)^{\frac{1}{2\sqrt{5}}} \geqslant V(t) > V(t_0) e^{-\frac{k}{2}},$$

As a result we obtain $\varepsilon(t) \geqslant a e^{-\frac{kt}{2}}$ $(a > 0)$, i.e. the function $\varepsilon(t)$ for $0 < \zeta \leqslant \varepsilon$ remains positive for all time until the encounter. If $\zeta > \varepsilon$, then, as in the case of (4.11), (4.12), we have $\varepsilon(t) \geqslant 0$ and, consequently, in the region $\zeta > \varepsilon$ the function $\varepsilon(t)$ does not decrease. Finally, using (4.15) and (4.24), it is obviously obtained that $\varepsilon(t)$ cannot decrease also in the case $\zeta = 0$.

Remark 1. Thus, we see that for theoretical purposes at least the choice of the regularizing function $R[\varepsilon, \zeta]$ is not unique. From a practical point of view, the most convenient would be the function

$$R[\varepsilon, \zeta] = \begin{cases} 1 & \text{for } \zeta > 0, \\ 0 & \text{for } \zeta = 0. \end{cases} \tag{4.25}$$

However, unfortunately, in this case the possibility arises of the occurrence of a sliding mode, analogous to that mentioned above for the $R$-strategy (4.11), (4.12), but leading to a rapid decrease in the value of $\varepsilon(t)$. Indeed, for $u_e = R[\varepsilon, \zeta] u_0$, where $R[\varepsilon, \zeta]$ is determined by formula (4.25), we have in the case $\zeta > 0$

$$\frac{d\zeta}{d\varepsilon} = \frac{\left( \frac{\omega^0}{\xi} + \delta v \right)}{\frac{\varepsilon(v + \zeta)}{2\mu\xi^2} \|\omega^0\|^2 + \frac{\|\delta v\|^2}{2v}} + o_3(\zeta), \tag{4.26}$$

$$o_3(\zeta) = 0 \text{ npu } \zeta = 0$$

and, for example, for $v = 0$ $\lim\limits_{z \to 0} \frac{d\zeta}{d\varepsilon} = - \frac{2\mu}{\varepsilon + \mu} < -1 - \gamma_1, -\gamma_1$, since $0 < \varepsilon < \mu - \gamma_2$ $(\gamma_1 > 0, \gamma_2 > 0)$. But at the same time for $\zeta = 0$ the value $d\zeta/d\varepsilon$ (4.19) for sufficiently small values of $\|v\|$ turns out to be arbitrarily close to $- -1$. Thus, if $v = \gamma = \text{const}$, where $\|\gamma\|$ is sufficiently small, then a peculiar sliding mode arises, and to able to lead even before the encounter to the elimination of the $\epsilon$-layer between the regions $G^{(1)}[y(\tau), \mu(\tau), \vartheta_e]$ and $G^{(2)}[z(\tau),$

Figure 13: Figure 13

$v(\tau)$, $\vartheta_z$]. The presence of the noted sliding mode is confirmed by direct numerical experiments on a digital computer.

*Note 2.* In the present article, the regularization of the pursuit problem is performed in such a way that the encounter problem continues to bear the character of a positional game. However, one can also go another way [1, 6]. Let us assume that the pursuer is guided the rule of extremal aiming $u = u_0[t]$ (1.19) until the attainability sets $G^{(1)}[y(\tau), \mu(\tau), \vartheta_0]$ (1.13) and $G^{(2)}[z(\tau), v(\tau), \vartheta_0]$ (1.14) do not coincide. Let us accept now, that at those moments of time $t = \tau$, when the attainability sets coincide, the choice of control $v(t)$, made by the second partner at the indicated moments of time, becomes known to the pursuing partner. This supposition, discriminating against the pursued object in the information respect, allows one to construct a strategy $u[t] = z(t), \mu(t), v(t), \vartheta_0]$ for the pursuer, ensuring an encounter no later than at the moment of absorption $\vartheta_0$. Such a strategy is constructed, for for example, from the condition of coincidence of the attainability sets during the entire time until the encounter, starting from that moment $t = t_*$, when $t_*$, when these sets comcided for the first time. The indicated condition leads in the final analysis to a conveniently realizable rule of copying and mirror reflection, analogous to that rule which was mentioned in work [6], (see § 4). The rule of copying and mirror reflection can be obtained not only under restrictions of the form (1.3), but also in a sufficiently general case of integral restrictions on the resources of controlling actions. However, a detailed analysis of this question goes beyond the scope of the present article.

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Figure 14: Figure 14