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MATHEMATICS

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Abstract

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MATHEMATICS

S. M. LOZINSKII

ESTIMATE OF THE DIFFERENCE OF TWO MATRICANTS

(Presented by Academician V. I. Smirnov on 25 IV 1966)

1. Notation. The word matrix means a square matrix of fixed order n, whose elements are complex numbers. If A is a matrix, then $[A]_{\mu\nu}$ $(\mu, \nu = 1, \dots, n)$ are its elements; $|A|$ denotes the matrix whose elements are the moduli of the corresponding elements of A; A^+ denotes the matrix whose diagonal (off-diagonal) elements are the real parts (moduli) of the corresponding elements of A; $|A|$ is called the modulus of the matrix A. The symbol E denotes the identity matrix; the symbol I denotes the matrix all of whose elements are equal to one. A matrix-function is a matrix whose elements are functions of the real parameter t. If $A = A(t)$ is a matrix-function, then

$$A^+(t) \stackrel{\text{def}}{=} (A(t))^+.$$

Inequalities between matrices, convergence of a sequence of matrices and of a matrix series, and also continuity and differentiability of matrix-functions are understood elementwise. The letters A, B, C, D, P, Q (also with indices) denote matrices or, if this is stipulated, matrix-functions; the latter are always assumed to be continuous. It is useful to note that

$$|A + B| \le |A| + |B|, \qquad |AB| \le |A|\,|B|.$$

A norm is introduced in the set of matrices, satisfying the three axioms of the norm of a linear complex space and the conditions

$$\|AB\| \le \|A\|\,\|B\|, \qquad \|E\| = 1.$$

It is known that

$$\lim_{k\to\infty} \|A_k - A\| = 0$$

is equivalent to

$$\lim_{k\to\infty} A_k = A$$

(see ([1]), Chap. 1, § 13, no. 3). The norm is used below only in the formulation and proof of Theorem 2 (paras. 4, 8, 10; see also paras. 12, 13).

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2. Consider the linear homogeneous system of differential equations (in vector notation)

$$dx/dt = A(t)x, \tag{1}$$

where $A(t)$ is a matrix-function on the interval $[t_0, T]$. The symbol

$${}^{t}_{\xi}\Omega A$$

(where $t_0 \le \xi,\ t \le T$) denotes the matricant of the matrix-function $A(t)$, i.e. the matrix-function whose columns are the solutions of system (1) satisfying the condition

$${}^{\xi}_{\xi}\Omega A = E$$

(for the properties of the matricant, see ([2]), Chap. XIV, §§ 5, 6). If, along with system (1), the system $dx/dt = B(t)x$ is also considered, then it is often necessary to estimate

$${}^{t}_{t_0}\Omega B - {}^{t}_{t_0}\Omega A.$$

Formulation of the results.

3. Theorem 1 (estimate of the modulus of the difference of matricants). *Let A, B, C, D be matrix-functions on $[t_0, T]$, and let*

$$A^{+}(t) \le C(t), \qquad |B(t) - A(t)| \le D(t) \quad \text{for } t_0 \le t \le T. \tag{2}$$

Then:

a)
$$\left|{}^{t}_{t_0}\Omega B - {}^{t}_{t_0}\Omega A\right| \le {}^{t}_{t_0}\Omega(C + D) - {}^{t}_{t_0}\Omega C \quad \text{for } t_0 \le t \le T.$$

b) If, moreover, the matrices $C(t_1), C(t_2), D(t_1), D(t_2)$ commute pairwise for $t_0 \leqslant t_1, t_2 \leqslant T$, then

$$\left|\Omega^{t}_{t_0} B - \Omega^{t}_{t_0} A\right| \leqslant \exp \int_{t_0}^{t} C(u)\, du \cdot \left\{ \exp \int_{t_0}^{t} D(u)\, du - E \right\} \quad \text{for } t_0 \leqslant t \leqslant T.$$

c) In particular, if

$$C(t) = q(t)I + \alpha(t)E, \qquad D(t) = \delta(t)I \quad \text{for } t_0 \leqslant t \leqslant T,$$

where the scalar functions $q(t), \alpha(t), \delta(t)$ are continuous on $[t_0, T]$, then

$$\left|\Omega^{t}_{t_0} B - \Omega^{t}_{t_0} A\right| \leqslant \exp \int_{t_0}^{t} \alpha\, du \cdot \exp\left(\int_{t_0}^{t} q\, du\, I\right) \cdot \left\{ \exp\left(\int_{t_0}^{t} \delta\, du\, I\right) - E \right\} \quad \text{for } t_0 \leqslant t \leqslant T.$$

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4. **Theorem 2 (estimate of the norm of the difference of multiplicative integrals).** Suppose there exist real functions $g(t), M(t), L(t), \delta(t)$, continuous on $[t_0, T]$ and satisfying the conditions

$$\left\|\Omega_\xi^t A\right\| \leqslant M(t)L(\xi)\exp\int_\xi^t g(u)\,du \quad \text{for } t_0 \leqslant \xi \leqslant t \leqslant T, \tag{3}$$

$$\|B(t) - A(t)\| \leqslant \delta(t) \quad \text{for } t_0 \leqslant t \leqslant T. \tag{4}$$

Then

$$\left\|\Omega_{t_0}^t B - \Omega_{t_0}^t A\right\| \leqslant M(t)L(t_0)\exp\int_{t_0}^t g\,du\cdot\left\{\exp\int_{t_0}^t ML\delta\,du - 1\right\} \quad \text{for } t_0 \leqslant t \leqslant T. \tag{5}$$

Proofs.

5. Suppose $|A_k| \leqslant P_k,\ |B_k - A_k| \leqslant D_k$ for $k = 1, \dots, N$. Then

$$|B_{NB_{N-1}} \cdots B_1 - A_{NA_{N-1}} \cdots A_1| \leqslant$$

$$\leqslant (P_N + D_N)(P_{N-1} + D_{N-1}) \cdots (P_1 + D_1) - P_{NP_{N-1}} \cdots P_1.$$

Proof. For $N = 1$ this is obvious. Noting that

$$B_{N+1}B_N \cdots B_1 - A_{N+1}A_N \cdots A_1 =$$

$$= (B_{N+1} - A_{N+1})B_N \cdots B_1 + A_{N+1}(B_N \cdots B_1 - A_N \cdots A_1),$$

we apply induction on N.

6. Suppose R is real, S is a real diagonal matrix, Δ is a nonnegative number, and

$$[R]_{\mu\mu} \geqslant -\frac{1}{2}, \qquad |[S]_{\mu\mu}| \leqslant \Delta \quad \text{for } \mu = 1, \dots, n;$$

$$[R]_{\mu\nu} \geqslant 0 \quad \text{for } \mu, \nu = 1, \dots, n;\ \mu \neq \nu.$$

Then

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$$|E + R + iS| \leqslant (1 + 2\Delta^2)(E + R).$$

Proof. By elementary calculations.

7. **Proof of Theorem 1.** Define the diagonal matrix-function S by the formulas

$$[S(t)]_{\mu\mu} \stackrel{\text{def}}{=} \operatorname{Im}[A(t)]_{\mu\mu}, \quad \text{for } \mu = 1, \dots, n.$$

Find a nonnegative number Δ and a natural number N_0 such that

$$|[S(t)]_{\mu\mu}| \leqslant \Delta, \qquad \frac{T - t_0}{N_0}[A^+(t)]_{\mu\mu} \geqslant -\frac{1}{2} \quad \text{for } \mu = 1, \dots, n;\ t_0 \leqslant t \leqslant T. \quad (6)$$

Fix $t,\ t_0 \leqslant t \leqslant T$, and a natural number $N > N_0$. Put

$$h \stackrel{\text{def}}{=} \frac{t - t_0}{N}, \qquad t_k \stackrel{\text{def}}{=} t_0 + kh \quad \text{for } k = 1, \dots, N. \quad (7)$$

Finally, set $A_k \stackrel{\text{def}}{=} A(t_k)$ for $k = 1, \dots, N$, and analogously for matrix-functions B, C, D, S. By item 6, (6), and (7),

$$\begin{gathered} |E + hA_k| = \left|E + hA_k^+ + ihS_k\right| \leq \\ \leq (1 + 2h^2\Delta^2)(E + hA_k^+) \leq (1 + 2h^2\Delta^2)(E + hC_k) \\ \text{for } k = 1, \dots, N. \end{gathered} \quad (8)$$

By (8) and the second of inequalities (2),

$$\begin{gathered} |E + hB_k| \leq |E + hA_k| + h\,|B_k - A_k| \leq |E + hA_k| + hD_k \leq \\ \leq (1 + 2h^2\Delta^2)(E + h(C_k + D_k)) \quad \text{for } k = 1, \dots, N. \end{gathered}$$

Replacing, in item 5, A_k by $E + hA_k$, B_k by $E + hB_k$, P_k by $(1 + 2h^2\Delta^2) \times (E + hC_k)$, and D_k by $(1 + 2h^2\Delta^2)hD_k$, we obtain

$$\begin{gathered} |(E + hB_N) \dots (E + hB_1) - (E + hA_N) \dots (E + hA_1)| \leq \\ \leq (1 + 2h^2\Delta^2)^N \{(E + h(C_N + D_N)) \dots (E + h(C_1 + D_1)) - \\ -(E + hC_N) \dots (E + hC_1)\}. \end{gathered}$$

Letting $N \to \infty$, we obtain a); b) follows from a) ((2), p.386); c) follows from b).

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8.

$$\left\|\Omega_\xi^t A\right\| \le \exp \int_\xi^t \|A(u)\|\, du \quad \text{for } t_0 \le \xi \le t \le T.$$

Proof. This follows from the representation of the matriciant as the limit of an integral product $((2), Ch.XIV, formula(49'))$.

9. Let A and B be matrix-functions on $[t_0, T]$. Define the matrix-function D by the formula

$$D(t) \stackrel{\text{def}}{=} B(t) - A(t) \quad \text{for } t_0 \le t \le T. \tag{9}$$

Then

$$\Omega_{t_0}^t B = \Omega_{t_0}^t A + \int_{t_0}^t \left(\Omega_\xi^t A\right) D(\xi) \left(\Omega_{t_0}^\xi B\right) d\xi \quad \text{for } t_0 \le t \le T. \tag{10}$$

Proof. Let $L(t)$ denote the left-hand side, and $\Pi(t)$ the right-hand side of formula (10). Then a simple calculation shows that the matrix-functions L and Π are solutions of the Cauchy problem

$$\frac{dX}{dt} = A(t)X + D(t)\Omega_{t_0}^t B, \quad X(t_0) = E \tag{11}$$

(X is the unknown matrix-function, $t_0 \le t \le T$). By uniqueness of the solution of problem (11), we have $L(t) = \Pi(t)$ for $t_0 \le t \le T$, as was required to prove.

10. **Proof of Theorem 2.** Keeping (9), define the matrix-functions I_k, I_k^* $(k = 0, 1, 2, ...)$ by the formulas

$$I_0(t) \stackrel{\text{def}}{=} \Omega_{t_0}^t A, \quad I_0^*(t) \stackrel{\text{def}}{=} \Omega_{t_0}^t B, \tag{12}$$

$$I_{k+1} \stackrel{\text{def}}{=} \int_{t_0}^t \left(\Omega_\xi^t A\right) D(\xi) I_k(\xi)\, d\xi \quad \text{for } k = 0, 1, 2, \dots, \tag{13}$$

$$I_{k+1}^* \stackrel{\text{def}}{=} \int_{t_0}^t \left(\Omega_\xi^t A\right) D(\xi) I_k^*(\xi)\, d\xi \quad \text{for } k = 0, 1, 2, \dots, \tag{14}$$

where $t_0 \le t \le T$. Then

$$\Omega_{t_0}^t B = I_0(t) + ... + I_k(t) + I_{k+1}^*(t) \quad \text{for } k = 0, 1, 2, \dots;\ t_0 \le t \le T.$$

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Indeed, for $k = 0$ this follows from (12), (14), and item 9; next we apply (using (10)) induction on k.

Denote by M some positive number such that

$$\left\|\Omega_{t_0}^t B\right\| \leq M \quad \text{for } t_0 \leq t \leq T. \tag{15}$$

Then

$$\left\|I_{k+1}^*(t)\right\| \leq M \exp \int_{t_0}^t \|A(u)\|\, du \cdot \frac{1}{(k+1)!} \left\{ \int_{t_0}^t \delta(u)\, du \right\}^{k+1} \tag{16}$$

for $k = 0, 1, 2, \ldots;\ t_0 \leq t \leq T$.

Indeed, for $k = 0$ inequality (16) follows from item 8, (15), (14), and (12). Next we apply induction on k. From (16) it follows that $\|I_{k+1}^*(t)\| \to 0$ as $k \to \infty,\ t_0 \leq t \leq T$, and consequently,

$$\Omega_{t_0}^t B = \sum_{k=0}^{\infty} I_k(t) \quad \text{for } k = 0, 1, 2, \ldots. \tag{17}$$

Let us now note that

$$\|I_k(t)\| \leq M(t) L(t_0) \exp \int_{t_0}^t g(u)\, du \cdot \frac{1}{k!} \left\{ \int_{t_0}^t ML\delta\, du \right\}^k \tag{18}$$

for $k = 0, 1, 2, \ldots;\ t_0 \leq t \leq T$.

Indeed, for $k = 0$ this is true by virtue of (12) and (3); we apply (using (13), (4), and (9)) induction on k. From (17) it follows that

$$\left\|\Omega_{t_0}^t B - \Omega_{t_0}^t A\right\| \leq \sum_{k=1}^{\infty} \|I_k(t)\| \quad \text{for } t_0 \leq t \leq T. \tag{19}$$

From (18) and (19) we obtain (5).

Additional Remarks

11. Let A and C be matrix-functions on $[t_0, T]$, with $A^+(t) \leq C(t)$ for $t_0 \leq t \leq T$. Then

$$\left|\Omega_{t_0}^t A\right| \leq \Omega_{t_0}^t C \quad \text{for } t_0 \leq t \leq T.$$

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Proof. With the notation of item 7, (8) is true, which gives

$$|(E + hA_N) \cdots (E + hA_1)| \leq (1 + 2h^2\Delta^2)^N (E + hC_N) \cdots (E + hC_1).$$

Let $N \to \infty$.

12. Theorem 1b) and item 11 generalize known results ([2], Ch. XIV, §6, VIII and §5, 5°; there—in different r
Condition (3) in Theorem 2 is fulfilled, for example, if $M(t) = L(t) = 1$, $g(t) = \gamma(A(t))$, where γ is the logarithmic norm of the matrix corresponding to the introduced matrix norm ([3], pp. 57–62).

13. The linear transformation $x = U(t)y$, where the matrix-function $U(t)$ has a continuous derivative and is nonsingular for $t_0 \leq t \leq T$, transforms system (1) into a system with matrix $Q(t)$, and

$$\Omega_\xi^t A = U(t) \left(\Omega_\xi^t Q\right) U^{-1}(\xi).$$

Hence it is easy to obtain a more general case of the fulfillment of (3) than in item 12.

14. If the matricant is defined as the sum of a series ([2], Ch. XIV, formula (10)), then in Theorems 1 and 2 it is sufficient, instead of continuity of the matrix-functions, to assume summability.

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[2] F. R. Gantmacher, *The Theory of Matrices*, 1953.
[3] S. M. Lozinskii, *Izv. Vyssh. Uchebn. Zaved., Matematika*, No. 5, 52 (1958).

Note: Figure translations are in progress. See original paper for figures.

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