ON THE THEORY OF ζ-FUNCTIONS OF THREE VARIABLES

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Abstract

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N. N. SANDAKOVA

ON THE THEORY OF ζ-FUNCTIONS OF THREE VARIABLES

(Presented by Academician S. L. Sobolev on 7 X 1966)

The formulation of the problem is described in the paper [1].

Let us denote

$$\zeta(A, m) = \sum_{x_i} \frac{1}{f^m(x_i)},$$

where $f(x_i)$ is a positive definite quadratic form in n variables, A is its matrix, $m > n/2$; the series is summed over all possible integral points.

In the present paper the following is proved.

Theorem. *The positive definite quadratic form*

$$f_0 = x^2 + y^2 + z^2 + xy + xz + yz,$$

corresponding to the densest packing of spheres in the space E^3, gives a local minimum of the function $\zeta(A, m)$ for $m \geq 2$.

Proof. Varying the form f_0, we obtain the form $f = f_0 + \bar{f}$, where

$$\bar{f} = \varepsilon_{11}x^2 + \varepsilon_{22}y^2 + \varepsilon_{33}z^2 + \varepsilon_{12}xy + \varepsilon_{13}xz + \varepsilon_{23}yz.$$

The form f_0 is transformed into itself by a group φ of integral unimodular transformations; the group φ, up to inversion, has order 24. As a fundamental domain Φ of the group φ we take the triangular angle determined by the inequalities $x - y > 0$, $y - z > 0$, $y + z > 0$. The initial point $B_1(x, y, z)$ will always be chosen in the closed domain Φ. The set K of integral points decomposes under the group φ into finitely many mutually disjoint sets of equivalent points. If the initial point $B_1(x, y, z) \in \Phi$, then, up to inversion, the set of points equivalent to it will be

$$B_1(x,y,z),\quad B_2(y,z,x),\quad B_3(z,x,y),\quad B_4(x,z,y),\quad B_5(y,x,z),\quad B_6(z,y,x),$$
$$B_7(x,y,-x-y-z),\quad B_8(y,-x-y-z,x),\quad B_9(-x-y-z,x,y),$$
$$B_{10}(x,-x-y-z,y),\quad B_{11}(y,x,-x-y-z),\quad B_{12}(-x-y-z,y,x),$$
$$B_{13}(-x-y-z,y,z),\quad B_{14}(y,z,-x-y-z),\quad B_{15}(z,-x-y-z,y),$$
$$B_{16}(-x-y-z,z,y),\quad B_{17}(y,-x-y-z,z),\quad B_{18}(z,y,-x-y-z),$$
$$B_{19}(x,-x-y-z,z),\quad B_{20}(-x-y-z,z,x),\quad B_{21}(z,x,-x-y-z),$$
$$B_{22}(x,z,-x-y-z),\quad B_{23}(-x-y-z,x,z),\quad B_{24}(z,-x-y-z,x);$$

denote it by ω_1. Depending on the choice on the boundary of Φ of the initial point $B_1(x,x,z)$, $B_1(x,y,y)$, $B_1(x,x,x)$, $B_1(x,x,-x)$, $B_1(x,x,0)$, $B_1(x,0,0)$, we respectively obtain the sets $\omega_2,\omega_3,\omega_4,\omega_5,\omega_6,\omega_7$, which are degenerate cases of the set ω_1. The set ω_i $(i=1,2,\dots,7)$ consists of $l_i=24/h$ points, where h is the order of the subgroup leaving the point $B_1(x,y,z)$ fixed. It is easy to compute

$$l_1=24,\quad l_2=12,\quad l_3=12,\quad l_4=4,\quad l_5=3,\quad l_6=12,\quad l_7=6.$$

Let B_{ik} be the k-th point of ω_i, $f_{ik}=f(B_{ik})$. We estimate the difference

$$\Delta\Sigma=\sum_{x,y,z}\frac{1}{f^m}-\sum_{x,y,z}\frac{1}{f_0^m}.$$

In view of the decomposition of the set K into the sets ω_i $(i=1,2,\dots,7)$, we have

$$\Delta\Sigma=\Delta_1\Sigma+\Delta_2\Sigma+\Delta_3\Sigma+\Delta_4\Sigma+\Delta_5\Sigma+\Delta_6\Sigma+\Delta_7\Sigma, \tag{1}$$

where

$$\Delta_i\Sigma=\sum_{x,y,z}\left(\frac{1}{f_{i1}^m}+\frac{1}{f_{i2}^m}+\dots+\frac{1}{f_{il_i}^m}-\frac{l_i}{f_0^m}\right),\quad B_1(x,y,z)\in\bar{\Phi}. \tag{1'}$$

Formulas (1) and $(1')$ make it possible to pass from summation over all possible integer points to summation only over those points which belong to the region $\overline{\Phi}$; therefore, in the further proof the problem is considered only in the region $\overline{\Phi}$.

In view of the group φ we have $f_0(B_{ik})=f_0$, $f_{ik}=f_0+\overline{f}_{ik}$, where $\overline{f}_{ik}=\overline{f}(B_{ik})$. Next one can choose $\varepsilon>0$ such that, for all $|\varepsilon_{lm}|<\varepsilon$ $(l=1,2,3;\ m=1,2,3)$, one has $\Delta_{ik}=\overline{f}_{ik}/f_0<1$; therefore, taking $1/f_0^m$ outside the parentheses and expanding $\Delta_i\Sigma$ by Newton' s binomial formula, we obtain the equality:

$$\Delta_i \Sigma = \sum_{x,y,z} \frac{1}{f_0^m} \left[-m(\Delta_{i1} + \Delta_{i2} + \cdots + \Delta_{ili}) + \frac{m(m+1)}{2} (\Delta_{i1}^2 + \Delta_{i2}^2 + \cdots + \Delta_{ili}^2) \right] + R(\varepsilon_{ik}^3). \tag{2}$$

As a result of the computations we obtain

$$\Delta_{i1} + \Delta_{i2} + \cdots + \Delta_{ili} = \frac{1}{6l_i} (3\varepsilon_{11} + 3\varepsilon_{22} + 3\varepsilon_{33} - \varepsilon_{12} - \varepsilon_{13} - \varepsilon_{23}), \tag{3}$$

$$\Delta_{i1}^2 + \Delta_{i2}^2 + \cdots + \Delta_{ili}^2 = \frac{1}{6l_i} \left[\beta + 2(\overline{\beta} - 2\beta) xyz(x+y+z)/f_0^2 \right], \tag{4}$$

where

$$\beta = 3\varepsilon_{11}^2 + 3\varepsilon_{22}^2 + 3\varepsilon_{33}^2 + \varepsilon_{12}^2 + \varepsilon_{13}^2 + \varepsilon_{23}^2 + 2\varepsilon_{11}\varepsilon_{22} + 2\varepsilon_{11}\varepsilon_{33} + 2\varepsilon_{22}\varepsilon_{33} - 2\varepsilon_{11}\varepsilon_{12} - 2\varepsilon_{11}\varepsilon_{13} - 2\varepsilon_{22}\varepsilon_{12} - 2\varepsilon_{22}\varepsilon_{23} - 2\varepsilon_{33}\varepsilon_{13} - 2\varepsilon$$

$$\overline{\beta} = 9\varepsilon_{11}^2 + 9\varepsilon_{22}^2 + 9\varepsilon_{33}^2 + \varepsilon_{12}^2 + \varepsilon_{13}^2 + \varepsilon_{23}^2 + 2\varepsilon_{11}\varepsilon_{22} + 2\varepsilon_{11}\varepsilon_{33} + 2\varepsilon_{22}\varepsilon_{33} - 6\varepsilon_{11}\varepsilon_{12} - 6\varepsilon_{11}\varepsilon_{13} - 6\varepsilon_{22}\varepsilon_{12} - 6\varepsilon_{22}\varepsilon_{23} - 6\varepsilon_{33}\varepsilon_{13} - 6\varepsilon$$

β is the sum of Δ_{ik}^2 at the points representing the minimum of the form f_0; β is positive definite, $\overline{\beta}$ is an indefinite form. From the equation $D = D_0 = \frac{1}{2}$ (D and D_0 are the determinants of the forms f and f_0) we find, restricting ourselves only to the quadratic part:

$$3\varepsilon_{11} + 3\varepsilon_{22} + 3\varepsilon_{33} - \varepsilon_{12} - \varepsilon_{13} - \varepsilon_{23} = \varepsilon_{12}^2 + \varepsilon_{13}^2 + \varepsilon_{23}^2 - 4\varepsilon_{11}\varepsilon_{22} - 4\varepsilon_{11}\varepsilon_{33} - 4\varepsilon_{22}\varepsilon_{33} + 2\varepsilon_{11}\varepsilon_{23} + 2\varepsilon_{22}\varepsilon_{13} + 2\varepsilon_{33}\varepsilon_{12} - \varepsilon_{12}\varepsilon_{13} - \varepsilon \tag{5}$$

Substituting the values (3) and (4) into (2), and then (2) into (1), and denoting

$$4 \sum_{x \neq y \neq z} \frac{1}{f_0^m} + 2 \sum_{x=y,\ z} \frac{1}{f_0^m} + 2 \sum_{x,\ y=z} \frac{1}{f_0^m} + \frac{2}{3} \sum_{x=y=z} \frac{1}{f_0^m} + \frac{1}{2} \sum_{x=y=-z} \frac{1}{f_0^m} + 2 \sum_{x=y,\ 0} \frac{1}{f_0^m} + \sum_{x,\ 0,\ 0} \frac{1}{f_0^m} = \theta_m,$$

and by S_m the same sum, but with the quantity $xyz(x+y+z)/f_0^{m+2}$ instead of $1/f_0^m$, we obtain

$$\Delta\Sigma = -m\mathcal{L}\theta_m + \frac{1}{2} m(m+1)\beta\theta_m + m(m+1)(\overline{\beta} - 2\beta)S_m. \tag{6}$$

$\Delta\Sigma$, in view of (5), is a completely determined quadratic form in the variables ε_{ik} and the parameters θ_m and S_m. Let us investigate this form.

$m \geq 4$. In view of (4), the inequality holds:

$$-\mathcal{L}\theta_m + \frac{1}{2}(m+1)\beta\theta_m + (m+1)(\overline{\beta} - 2\beta)S_m > -\mathcal{L}\theta_m + \frac{1}{2}(m+1)\beta. \qquad (7)$$

In order not to interrupt the sequence of the exposition, the estimates of θ_m and S_m will be given later; for now we shall use the ready result

$$1 < \theta_4 < 1.2. \qquad (8)$$

For $m \geq 4$ and condition (8), the right-hand side of inequality (7) will always be positive definite, while for $m = 2, 3$ it will already be an indefinite form. For $m \geq 4$ the theorem is proved.

For $m = 2, 3$ the estimates hold

$$2.109 < \theta_2 < 2.3, \qquad -0.111 < S_2 < -0.067; \qquad (9)$$

$$1.14 < \theta_3 < 1.4, \qquad -0.04 < S_3 < 0. \qquad (10)$$

The variations ε_{ik} are not arbitrary, but satisfy the equation $D = 1/2$, from which we find

$$\varepsilon_{11} = (\varepsilon_{12} + \varepsilon_{13} + \varepsilon_{23} - 3\varepsilon_{22} - 3\varepsilon_{33})/3 \qquad (11)$$

(we take into account only the linear part, since the quadratic part, when substituted into (6), already gives the fourth degree in ε_{ik}). Substituting expression (11) into (6), we obtain

$$\begin{aligned}\Delta\Sigma = 6[(m-1)\theta_m + 4(m+1)S_m](\varepsilon_{22}^2 + \varepsilon_{33}^2) + [(m-2)\theta_m - \\ - 4(m+1)S_m](\varepsilon_{12}^2 + \varepsilon_{13}^2) + (2m-3)\theta_m\varepsilon_{23}^2 + 6[(m-1)\theta_m + \\ + 4(m+1)S_m]\varepsilon_{22}\varepsilon_{33} - 2[(m-1)\theta_m + 4(m+1)S_m](\varepsilon_{22}\varepsilon_{12} + \varepsilon_{33}\varepsilon_{13}) + \\ + [(m-1)\theta_m + 4(m+1)S_m](\varepsilon_{22}\varepsilon_{13} + \varepsilon_{33}\varepsilon_{12}) - 5[(m-1)\theta_m + \\ + 4(m+1)S_m](\varepsilon_{22}\varepsilon_{23} + \varepsilon_{33}\varepsilon_{23}) + [(2-m)\theta_m + 4(m+1)S_m]\varepsilon_{12}\varepsilon_{13} + \\ + [\theta_m + 8(m+1)S_m](\varepsilon_{12}\varepsilon_{23} + \varepsilon_{13}\varepsilon_{23})\end{aligned}$$

a quadratic form in five variables.

The principal minors of the determinant of this form will be

$$D_1 = C_1[(m-1)\theta_m + 4(m+1)S_m],$$

$$D_2 = C_2[(m-1)\theta_m + 4(m+1)S_m]^2,$$

$$D_3 = C_3(11m-29)\left[\theta_m + \frac{100(m+1)}{(29-11m)}S_m\right],$$

$$D_4 = C_4\left[\frac{3}{2}(m-3)\theta_m - 18(m+1)S_m\right][(17m-35)\theta_m - 76(m+1)S_m],$$

$$D_5 = C_5\left[\frac{3}{2}(m-3)\theta_m - 18(m+1)S_m\right]\left[(m^2-4m+3)\theta_m^2 - 8m(m+1)\theta_m S_m - 48(m+1)^2 S_m^2\right],$$

where $C_i > 0$ $(i = 1, 2, 3, 4, 5)$ are constants.

It is easy to verify for $m = 2, 3$ and under conditions (9) and (10) that the principal minors are positive; consequently, $\Delta\Sigma$ is a positive definite form, which proves the theorem. From this there also follows once more the proof for $m \geqslant 4$, since for $\theta_m > 1$ and $S_m < 0$ the D_i $(i = 1, 2, 3, 4, 5)$ are positive.

Estimations of θ_m and S_m. The equalities are obvious

$$\sum_{x=y=z} \frac{1}{f_0^m} = \frac{1}{6^m}\zeta(2m), \qquad \sum_{x=y=-z} \frac{1}{f_0^m} = \frac{1}{2^m}\zeta(2m), \tag{12}$$

$$\sum_{x=y,\,0} \frac{1}{f_0^m} = \frac{1}{3^m}\zeta(2m), \qquad \sum_{x,\,0,\,0} \frac{1}{f_0^m} = \zeta(2m).$$

Let the sums

$$\sum_{x\neq y\neq z} \frac{1}{f_0^m}, \qquad \sum_{x=y,\,z} \frac{1}{f_0^m}, \qquad \sum_{x,\,y=z} \frac{1}{f_0^m}, \tag{13}$$

$$\sum_{x=-y=z} \frac{xyz(x+y+z)}{f_0^{m+2}}, \qquad \sum_{x=-y,\,z} \frac{xyz(x+y+z)}{f_0^{m+2}}, \qquad \sum \frac{xyz(x+y+z)}{f_0^{m+2}} \tag{14}$$

be computed for all points for which $f_0 \leqslant c^2$. The sums of the series (13) and (14) for which $f_0 > c^2$ will be estimated with the aid of an integral.

In the space E^4, in Cartesian coordinates, consider the surface $t = 1/f_0^m$. In the space E^3 the set K forms a cubic lattice Γ. The region enclosed between Φ and the corresponding part of the surface is filled by prisms whose bases are the cubes of the lattice Γ, and whose height is the smallest value taken by the function $1/f_0^m$ in the given cube. The step body thus obtained lies beneath the surface. To each point $B_1(x,y,z)$ of the lattice Γ we assign the cube Π with vertices $B_1(x,y,z)$, $B_2(x-1,y,z)$, $B_3(x,y-1,z)$, $B_4(x-1,y-1,z)$, $B_5(x,y,z-1)$, $B_6(x-1,y,z-1)$, $B_7(x,y-1,z-1)$, $B_8(x-1,y-1,z-1)$.

Under this correspondence: 1) the value $1/f_0^m$ will be the smallest in the cube Π; 2) for all points $B_1(x,y,z)\in\Phi$, with the exception of the plane $y+z=1$, the cubes Π fall completely into the region $\overline{\Phi}$. We reason similarly for points lying on the boundary of Φ (here, instead of cubes there will be squares). Comparing the volumes of the step body in the region lying under the surface $1/f_0^m$, and taking into account that

$$1/f_0^m(x,y,-y+1) < 1/f_0^m(x,y,-y),$$

we obtain, for $f_0 \geq c^2$, the estimates

$$\sum_{x\neq y\neq z}\frac{1}{f_0^m} < \frac{\pi\sqrt{2}}{12(2m-3)(c-\sqrt{6})^{2m-3}} + \frac{\pi}{2(2m-2)(c-\sqrt{2})^{2m-2}}, \tag{15}$$

$$\sum_{x=y,\ z}\frac{1}{f_0^m} < \frac{14\pi}{45\sqrt{2}(2m-2)(c-\sqrt{6})^{2m-2}}, \qquad \sum_{x,\ y=z}\frac{1}{f_0^m} < \frac{\pi}{5\sqrt{2}(2m-2)(c-\sqrt{6})^{2m-2}}.$$

The following inequalities hold:

$$\begin{aligned} &xyz(x+y+z) < \frac{1}{3}f_0^2, \\ &xyz(x+y+z) \leq \frac{1}{7}f_0^2, \qquad \text{if } z>0, \\ &xyz(x+y+z) \leq \frac{1}{12}f_0^2, \qquad \text{if } y=z. \end{aligned} \tag{16}$$

The values of the sums (13) and (14) for $m=2$ and $f_0 \leq 225$ were computed on a computer by L. Voitishchek of Novosibirsk; they are respectively equal to

$$0.148072\ldots,\quad 0.028883\ldots,\quad 0.017752\ldots,\quad -0.012944\ldots,\quad -0.003021\ldots,\quad 0.000978\ldots \tag{17}$$

Taking into account the values (17) and computing (12) and (15), we obtain the estimate (9) for θ_2. Using the values (17) and the inequalities (16), and then the estimates (15), we obtain an estimate for S_2. Knowing the estimates for θ_2 and S_2, we estimate θ_3 and S_3.

As a result of computing (12) we obtain the lower estimate (10) for θ_3. Next we note that, among all points for which the values $1/f_0^m$ do not enter into (12), the point $B_1(2,1,-1)$ gives the smallest value $f_0=5$; hence we easily obtain the upper estimate (10) for θ_3. The estimates for S_3 and θ_4 are computed analogously.

In conclusion I express my gratitude to L. Voitishchek for computing the sums.

Mathematical Institute named after V. A. Steklov
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1. B. N. Delone, N. N. Sandakova, S. S. Ryshkov, DAN, **162**, No. 6 (1965)

Note: Figure translations are in progress. See original paper for figures.