# CONTINUOUS LINEAR FORMS

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## Abstract

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*MATHEMATICS*

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# CONTINUOUS LINEAR FORMS

# ON INDUCTIVE LIMITS OF DERIVED STRAIGHTS

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Let $(A_\iota, \varphi_{\iota\chi})$ be a projective system of sets with respect to a partially ordered set of indices $I$, filtered to the right. For each $\iota \in I$ let $\mathbf{R}^{A_\iota}$ be the vector space of all real-valued functions on the set $A_\iota$, endowed with the topology of pointwise convergence; for each pair of indices $\iota \leq \chi$ let $f_{\chi\iota} : \mathbf{R}^{A_\iota} \to \mathbf{R}^{A_\chi}$ be the mapping assigning to each function $x_\iota$ on $A_\iota$ the function $x_\chi = x_\iota \circ \varphi_{\iota\chi}$ on $A_\chi$. Then the topological vector spaces $\mathbf{R}^{A_\iota}$ $(\iota \in I)$ and the continuous linear mappings $f_{\chi\iota}$ $(\iota \leq \chi)$ form an inductive system; the purpose of the present work is the study of continuous linear forms on the topological vector space

$$F = \varinjlim \mathbf{R}^{A_\iota},$$

which is the (locally convex) inductive limit of the system $(\mathbf{R}^{A_\iota}, f_{\chi\iota})$.

Let $\mathfrak{A}_\iota = \mathfrak{P}(A_\iota)$ for each $\iota \in I$ be the set of all parts of the set $A_\iota$, and let $\psi_{\chi\iota}$, for each pair of indices $\iota \leq \chi$, be the mapping $\mathfrak{A}_\iota \to \mathfrak{A}_\chi$ assigning to each part of the set $A_\iota$ its inverse image under the mapping $\varphi_{\iota\chi}$; the sets $\mathfrak{A}_\iota$ $(\iota \in I)$ and the mappings $\psi_{\chi\iota}$ $(\iota \leq \chi)$ form an inductive system. A subset $\mathfrak{M}$ of the set

$$\mathfrak{A} = \varinjlim \mathfrak{A}_\iota,$$

which is the inductive limit of the system $(\mathfrak{A}_\iota, \psi_{\chi\iota})$, will be called **final** if, for each $\iota \in I$, first, there exists only a finite number of elements in $\mathfrak{M}$ having representatives in $\mathfrak{A}_\iota$, and, second, the representatives of these elements can be chosen pairwise disjoint in $A_\iota$.

Let $m$ be an arbitrary element of the set $\mathfrak{A}$ having a representative $M_\iota \subset A_\iota$ in $\mathfrak{A}_\iota$ for each $\iota \geq \iota_0 = \iota_0(m)$. Then the topological vector spaces $\mathbf{R}^{M_\iota}$ $(\iota \geq \iota_0)$ and the continuous linear mappings $\mathbf{R}^{M_\iota} \to \mathbf{R}^{M_\chi}$ (each of which is defined for $\iota_0 \leq \iota \leq \chi$ and carries a function $x_\iota$ on $M_\iota$ to the function $x_\chi = x_\iota \circ \varphi_{\iota\chi}$ on $M_\chi$) form an inductive system. For each $\iota \geq \iota_0$ define a continuous linear mapping

$$i_{m\iota} : \mathbf{R}^{M_\iota} \to \mathbf{R}^{A_\iota},$$

which assigns to each function $x_\iota$ on $M_\iota$ the function on $A_\iota$ that coincides with $x_\iota$ on $M_\iota$ and is identically equal to zero outside $M_\iota$. Let

$$F_m = \varinjlim \mathbf{R}^{M_\iota}$$

be the inductive limit of the system of spaces $\mathbf{R}^{M_\iota}$ $(\iota \geq \iota_0)$ and mappings $\mathbf{R}^{M_\iota} \to \mathbf{R}^{M_\chi}$ $(\iota \leq \chi)$; since, for $\iota \leq \chi$, the composition $f_{\chi\iota} \circ i_{m\iota}$ coincides with the composition of the mappings $\mathbf{R}^{M_\iota} \to \mathbf{R}^{M_\chi}$ and $i_{m\chi}$, there is defined an (injective) continuous linear mapping

$$i_m : F_m \to F,$$

which assigns to each element $x \in F_m$, having representative $x_\iota$ in $\mathbf{R}^{M_\iota}$, the element of $F$ that is the canonical image of the element $i_{m\iota}(x_\iota) \in \mathbf{R}^{A_\iota}$.

Let

$$A = \varprojlim A_\iota$$

be the projective limit of the system $(A_\iota, \varphi_{\iota\chi})$, and let $\varphi_\iota : A \to A_\iota$ $(\iota \in I)$ be the canonical mappings. For each $\iota \in I$ define a continuous linear mapping $h_\iota : \mathbf{R}^{A_\iota} \to \mathbf{R}^A$, assigning to each function $x_\iota$ on $A_\iota$ the function $x = x_\iota \circ \varphi_\iota$ on $A$. Since $h_\iota(x_\iota) = h_\chi(x_\chi)$, if the canonical images of the elements $x_\iota \in \mathbf{R}^{A_\iota}$ and $x_\chi \in \mathbf{R}^{A_\chi}$

coincide in $F$, and hence a continuous linear mapping has been defined,

$$h : F \to \mathbf{R}^A,$$

which assigns to each element $x \in F$ the function $x_\iota \circ \varphi_\iota$ on $A$, where $x_\iota$ is a representative of the element $x$ in $\mathbf{R}^{A_\iota}$. The main result of the paper is the following

**Theorem.** A continuous linear form $s$ on $F$ is representable in the form

$$s = t \circ h,$$

where $t$ is some continuous linear form on $\mathbf{R}^A$ (uniquely determined by the form $s$), if and only if, whatever finite subset $\mathfrak{M}$ of $\mathfrak{A}$ is taken,

$$s \circ i_{\mathfrak{m}} = 0$$

for all but, possibly, finitely many elements $\mathfrak{m} \in \mathfrak{M}$.

**Proof.** Let $s$ be an arbitrary continuous linear form on $F$, and let $f_\iota$, for each $\iota \in I$, be the canonical continuous linear mapping $\mathbf{R}^{A_\iota} \to F$. Then, for each $\iota \in I$, the composition $s_\iota = s \circ f_\iota$ is uniquely representable in the form

$$x_\iota \to \sum_{\alpha \in A_\iota} x_\iota(\alpha) s_\iota(\alpha),$$

where $\alpha \to s_\iota(\alpha)$ is a numerical function on $A_\iota$, for which the set $\operatorname{supp}(s_\iota)$ of those $\alpha \in A_\iota$ such that $s_\iota(\alpha) \neq 0$ is finite; moreover,

$$s_\iota(\alpha) = \sum_{\varphi_{\iota\chi}(\beta) = \alpha} s_\chi(\beta)$$

for any pair of indices $\iota \leq \chi$ and every $\alpha \in A_\iota$. In particular,

$$\operatorname{supp}(s_\iota) \subset \varphi_{\iota\chi}(\operatorname{supp}(s_\chi)) \qquad (\iota \leq \chi).$$

Let $\mathfrak{M}$ be an arbitrary finite subset of $\mathfrak{A}$; suppose that $\mathfrak{M}$ is infinite and that $s \circ i_{\mathfrak{m}} \neq 0$ for every $\mathfrak{m} \in \mathfrak{M}$. Then one can choose indices $\iota_1 < \iota_2 < \cdots$ so that, for each integer $n > 0$, the set $\mathfrak{M}$ contains at least $n$ elements having pairwise disjoint representatives $M_{n,1}, \dots, M_{n,n}$ in $A_{\iota_n}$, for which

$$\operatorname{supp}(s_{\iota_n}) \cap M_{n,k} \neq \emptyset \qquad (n = 1, 2, \dots;\ k = 1, \dots, n).$$

Let $n(s_\iota)$, for each $\iota \in I$, be the number of elements of the set $\operatorname{supp}(s_\iota)$; then $n(s_{\iota_k}) \geq k$ for every $k = 1, 2, \dots$, and, consequently, the family of numbers $n(s_\iota)$ $(\iota \in I)$ is unbounded. On the other hand, if $s = t \circ h$, where $t$ is some continuous linear form on $\mathbf{R}^A$, then, for each $x \in F$,

$$s(x) = \sum_{\alpha \in A} y(\alpha) t(\alpha),$$

where $y = h(x)$, and $\alpha \to t(\alpha)$ is a numerical function on $A$, nonzero only on some finite set $\operatorname{supp}(t)$. In this case

$$\operatorname{supp}(s_\iota) \subset \varphi_\iota(\operatorname{supp}(t))$$

for every $\iota \in I$; consequently, the numbers $n(s_\iota)$ $(\iota \in I)$ are bounded above by the number of elements of the set $\operatorname{supp}(t)$.

Conversely, suppose that $s$ is a continuous linear form on $F$ not representable in the form $s = t \circ h$. Then the numbers $n(s_\iota)$ $(\iota \in I)$ cannot be bounded in the aggregate. Indeed, in the contrary case the family of these numbers stabilizes, since $n(s_\iota) \leq n(s_\chi)$ for $\iota \leq \chi$. Replacing, if necessary, the set $I$ by a cofinal subset, one may assume that $\varphi_{\iota\chi}$ maps $\operatorname{supp}(s_\chi)$ bijectively onto $\operatorname{supp}(s_\iota)$, whatever $\iota \leq \chi$. Then there exists in $A$ a finite set $M$ for which

$$\varphi_\iota(M) = \operatorname{supp}(s_\iota)$$

for every $\iota \in I$; since, moreover, $s_\iota(\alpha) = s_\chi(\beta)$ if $\alpha = \varphi_{\iota\chi}(\beta)$ and $\beta \in \operatorname{supp}(s_\chi)$, the continuous linear form on $\mathbf{R}^A$

$$x \to t(x) = \sum_{\alpha \in A} x(\alpha) t(\alpha),$$

where $t(\alpha) = s_\iota(\varphi_\iota(\alpha))$ for $\alpha \in M$ and $t(\alpha) = 0$ for $\alpha \notin M$, is such that $s = t \circ h$, and the latter representation is unique, since the image of the mapping $h$ is everywhere dense in $\mathbf{R}^A$.

Thus, one may suppose that the numbers $n(s_\iota)$ $(\iota \in I)$ are not bounded in the aggregate. We construct by induction a sequence of indices $\iota_1 < \iota_2 < \cdots$ and a sequence of elements $\alpha_n \in A_{\iota_n}$ $(n = 1, 2, \ldots)$ in the following way. Choose the index $\iota_1$ and the element $\alpha_1 \in \operatorname{supp}(s_{\iota_1})$ so that $n(s_{\iota_1}) \geq 2$ and the number of elements of the set $\operatorname{supp}(s_\iota) \setminus \varphi^{-1}_{\iota_1\iota}(\alpha_1)$ is not bounded for $\iota > \iota_1$. If the indices $\iota_1 < \cdots < \iota_{n-1}$ and the elements $\alpha_1, \ldots, \alpha_{n-1}$ have already been chosen, then we determine the index $\iota_n$ and the element $\alpha_n \in \operatorname{supp}(s_{\iota_n})$ so that the number of elements of the set

$$\operatorname{supp}(s_{\iota_n}) \setminus \bigcup_{m<n} \varphi^{-1}_{\iota_m\iota_n}(\alpha_m)$$

is $\geq 2$, $\varphi_{\iota_m\iota_n}(\alpha_n) \neq \alpha_m$ for $m < n$, and so that the number of elements of the set

$$\operatorname{supp}(s_\iota) \setminus \bigcup_{m\leq n} \varphi^{-1}_{\iota_m\iota}(\alpha_m)$$

is not bounded for $\iota > \iota_n$. Let $\mathfrak{m}_n$, for each $n = 1, 2, \ldots$, be the canonical image in $\mathfrak{A}$ of the one-point set $\{\alpha_n\} \in \mathfrak{A}_{\iota_n}$; since $\alpha_n \in \operatorname{supp}(s_{\iota_n})$, the form $s$ is such that $s \circ i_{\mathfrak{m}_n} \neq 0$ for every $n = 1, 2, \ldots$. According to the assumptions made, the set $\mathfrak{M} = \{\mathfrak{m}_1, \ldots\}$ cannot be final; therefore there exists $\iota \in I$ such that all elements of some infinite part of the set $\mathfrak{M}$ have representatives in $\mathfrak{A}_\iota$. We shall suppose that, for every $n = 1, 2, \ldots$, the element $\mathfrak{m}_n \in \mathfrak{A}$ has a representative $M_n$ in $\mathfrak{A}_\iota$. Since $\varphi_{\iota_m\iota_n}(\alpha_n) \neq \alpha_m$ for $m < n$, one may suppose that the sets $M_n$ $(n = 1, 2, \ldots)$ are pairwise disjoint. For every $n = 1, 2, \ldots$ define a function $x_n$ on $A_{\iota_n}$ by putting $x_n(\alpha) = 0$ (for $\alpha \in A_{\iota_n} \setminus \alpha_n$) and $x_n(\alpha_n) = n/s_{\iota_n}(\alpha_n)$. Then $s(f_{\iota_n}(x_n)) = n$, for every $n = 1, 2, \ldots$; consequently, the form $s$ is not bounded on the sequence of elements $f_{\iota_n}(x_n) \in F$ $(n = 1, 2, \ldots)$. On the other hand, $f_{\iota_n}(x_n) = f_\iota(y_n)$, where $y_n$ is a function on $A_\iota$ which takes the value $n/s_{\iota_n}(\alpha_n)$ on $M_n$ and is equal to zero on $A_\iota \setminus M_n$. Since the sets $M_n$ $(n = 1, 2, \ldots)$ are pairwise disjoint, the functions $y_n$ $(n = 1, 2, \ldots)$ form a bounded sequence in $R^{A_\iota}$; hence the sequence of elements $f_\iota(y_n)$ is bounded in $F$. Since the form $s$ is not bounded on the elements $f_\iota(y_n)$ $(n = 1, 2, \ldots)$, it cannot be continuous on $F$; thus the theorem is proved.

**Corollary.** *Every continuous linear form $s$ on $F$ is uniquely representable in the form $s = t \circ h$, where $t$ is some continuous linear form on $\mathbf{R}^A$, if every final subset in $\mathfrak{A}$ is finite.*

Let $\mathbf{R}^{(A_\iota)}$, for every $\iota \in I$, be the vector space of all real numerical functions $s_\iota$, each of which is defined on $A_\iota$ and is equal to zero everywhere outside some (depending on $s_\iota$) finite set $\operatorname{supp}(s_\iota)$. For each pair of indices $\iota \leq \varkappa$ let $g_{\iota\varkappa} : \mathbf{R}^{(A_\varkappa)} \to \mathbf{R}^{(A_\iota)}$ be the mapping which assigns to each function $s_\varkappa \in \mathbf{R}^{(A_\varkappa)}$ the function

$$\alpha \to s_\iota(\alpha) = \sum_{\varphi_{\iota\varkappa}(\beta)=\alpha} s_\varkappa(\beta),$$

belonging to the space $\mathbf{R}^{(A_\iota)}$. The vector spaces $\mathbf{R}^{(A_\iota)}$ $(\iota \in I)$ and the linear mappings $g_{\iota\varkappa}$ $(\iota \leq \varkappa)$ form a projective system. The projective limit

$$G = \varprojlim \mathbf{R}^{(A_\iota)}$$

of the system $(\mathbf{R}^{(A_\iota)}, g_{\iota\varkappa})$ will be identified with the vector space $F'$, conjugate to $F$.

For every $\iota \in I$ a linear mapping $k_\iota : \mathbf{R}^{(A)} \to \mathbf{R}^{(A_\iota)}$ is defined, assigning to each element $s \in \mathbf{R}^{(A)}$ the function

$$\alpha \to s_\iota(\alpha) = \sum_{\varphi_\iota(\beta)=\alpha} s(\beta),$$

belonging to the space $\mathbf{R}^{(A_\iota)}$. Since $k_\iota = g_{\iota\kappa} \circ k_\kappa$ for all $\iota \leq \kappa$, a linear mapping is defined,

$$k : \mathbf{R}^{(A)} \to G,$$

which assigns to each element $s \in \mathbf{R}^{(A)}$ the element of $G$ whose canonical projection in $\mathbf{R}^{(A_\iota)}$ is equal to $k_\iota(s)$, for each $\iota \in I$. Since the image of the mapping $h$ is everywhere dense in $\mathbf{R}^A$, the mapping $k$ is injective.

Let an element $\mathfrak{m} \in \mathfrak{A}$ have, for each $\iota \geq \iota_0$, a representative $M_\iota$ in $\mathfrak{A}_\iota$. Then the vector spaces $\mathbf{R}^{(M_\iota)}$ $(\iota \geq \iota_0)$ and the naturally defined linear mappings $\mathbf{R}^{(M_\kappa)} \to \mathbf{R}^{(M_\iota)}$ $(\iota \leq \kappa)$ form a projective system (a projective spectrum), whose limit we denote by $G_{\mathfrak{m}}$. For each $\iota \geq \iota_0$ define the restriction mapping $j_{m_\iota} : \mathbf{R}^{(A_\iota)} \to \mathbf{R}^{(M_\iota)}$; since for every pair of indices $\iota \leq \kappa$ the composition $j_{m_\iota} \circ g_{\iota\kappa}$ coincides with the composition of the mappings $j_{m_\kappa}$ and $\mathbf{R}^{(M_\kappa)} \to \mathbf{R}^{(M_\iota)}$, a (contractive) linear mapping is thereby defined,

$$j_{\mathfrak{m}} : G \to G_{\mathfrak{m}},$$

which assigns to each element $s \in G$ the element $j_{\mathfrak{m}}(s) \in G_{\mathfrak{m}}$ whose canonical image in $\mathbf{R}^{(M_\iota)}$ is the restriction to $M_\iota$ of the canonical image of the element $s$ in $\mathbf{R}^{(A_\iota)}$, for each $\iota \geq \iota_0$.

The theorem proved in the present paper is equivalent to the following assertion:

*The image of the mapping $k$ consists of those and only those elements $s \in G$ for which, whatever final set $\mathfrak{M} \subset \mathfrak{A}$ may be, $j_{\mathfrak{m}}(s) = 0$ for every $\mathfrak{m} \in \mathfrak{M}$, except possibly for a finite number of them. In particular, the mapping $k$ is an isomorphism of the space $\mathbf{R}^{(A)}$ onto $G$, if every final subset in $\mathfrak{A}$ is finite.*

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## CITED LITERATURE

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2. A. Douady, *C. R.*, **259**, 2946 (1964).

*Note: Figure translations are in progress. See original paper for figures.*

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