



---

Soviet-era science, translated into English

# LOCAL RELAXATION THEORY FOR NONLINEAR EQUATIONS

1967

SovietRxiv

---

View the original and related papers at <https://sovietrxiv.org/items/ru-196701.86438>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

**Abstract**

**Full Text**

UDC 517.948:518

**MATHEMATICS**

**G. D. MAISTROVSKII**

## **LOCAL RELAXATION THEORY FOR NON-LINEAR EQUATIONS**

*(Presented by Academician L. V. Kantorovich on 9 I 1967)*

Consider, in a real Hilbert space, the equation

$$Fx = 0 \tag{1}$$

with potential operator  $F$ :  $Fx = \text{grad } \varphi(x)$ , where  $\varphi$  is some functional. To solve equation (1), relaxation methods are usually applied, leading to minimization of the functional  $\varphi$ .

In the work of Yu. I. Lyubich <sup>(1)</sup>, a general theory of relaxation processes was developed for the case of a quadratic functional. The analogous theory in the general case must evidently be local. Its construction is the aim of the present note.

Suppose that, in a neighborhood of some solution of equation (1), the functional  $\varphi(x)$  has linear Gateaux differentials of the 1st and 2nd orders\*. Then Taylor's formula is valid:

$$\begin{aligned} \varphi(x) = \varphi(a) + (Fa, x - a) + \frac{1}{2}(F'(a + \xi(x - a))(x - a), x - a) \\ (0 < \xi < 1), \end{aligned} \tag{2}$$

where  $F'(z)$  is the Gateaux derivative of the operator  $F$  at the point  $z$ . Suppose further that the second differential  $D^2\varphi(x, v, v) = (F'(x)v, v)$  is a positive-definite bounded quadratic functional

$$m\|v\|^2 \leq (F'(x)v, v) \leq M\|v\|^2.$$

A sequence  $\{x_k\}_0^\infty \subset H$  is called a **relaxation sequence** (r.s.) if  $\varphi(x_{k+1}) \leq \varphi(x_k)$  ( $k = 0, 1, 2, \dots$ ). We shall call an r.s. **well localized** (w.l.r.s.) if all its terms belong to that neighborhood  $S(x^0, r) = \{x : \|x - x^0\| \leq r\}$ , in which

the above conditions are fulfilled. In particular, for  $r = \infty$ , every r.s. is well localized.

Let  $\{x_k\}_0^\infty$  be a w.l.r.s. Define the vectors  $y_k$  by setting

$$x_{k+1} = x_k - a_k y_k, \quad (3)$$

where  $a_k = \|x_{k+1} - x_k\|$ . Next introduce the relaxation multipliers  $q_k$  and the relaxation angles  $\theta_k$  by the equalities\*\* (cf. with (1))

$$q_k = 2 + 2\Delta\varphi(x_k)/a_k(Fx_k, y_k), \quad (Fx_k, y_k) = \|F(x_k)\| \cos \theta_k,$$

where  $\theta\varphi(x_k) = \varphi(x_{k+1}) - \varphi(x_k)$ . The relaxationality of the sequence  $\{x_k\}_0^\infty$  is equivalent to the inequality  $0 \leq q_k \leq 2$ .

It can be shown that the representation

$$\Delta\varphi(x_k) = -q_k(2 - q_k) \cos^2 \theta_k \Delta_k h_k, \quad (4)$$

\* Regarding differential calculus for functionals and operators, see, for example, (2).

\*\* We take  $q_k = 1$  when  $(Fx_k, y_k) = 0$ , and  $q_k = 0$  when  $a_k = 0$ ,  $(Fx_k, y_k) \neq 0$ ; the angle  $\theta_k$  is chosen in  $[0, \pi/2]$ , with  $\theta_k = 0$  if  $F(x_k) = 0$ .

where  $\Delta_k = \varphi(x_k) - \varphi(x^0)$ ,

$$h_k = \frac{1}{2} \|F(x_k)\|^2 (\Delta_k (F'(x_k - \alpha_k \xi_k y_k) y_k, y_k))^{-1}$$

( $0 < \zeta_k < 1$ ), and for the quantities  $h_k$  the estimates \* hold

$$h^{-1} \leq h_k \leq h, \quad (5)$$

where  $h = Mm^{-1}$  is the measure of conditioning. From formula (4) and inequality (5) there follow, exactly as in (1), the following results.

**Theorem 1.** For convergence of the l.r.p. to the solution  $x^0$  it is necessary and sufficient that the series

$$\sum_{k=0}^{\infty} q_k(2 - q_k) \cos^2 \theta_k$$

diverge.

**Theorem 2.** Put

$$\varkappa = \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \frac{\Delta_{k+1}}{\Delta_k}, \quad \sigma = \underline{\lim}_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} q_k (2 - q_k) \cos^2 \theta_k.$$

For any l.r.p. the inequality holds

$$1 - h\sigma \leq \varkappa \leq 1 - h^{-1}\sigma. \quad (6)$$

An l.r.p. is called **normally convergent** if  $\varkappa < 1$ . From normal convergence there obviously follows convergence to the solution  $x^0$  with the rate of a geometric progression.

**Theorem 3.** An l.r.p. converges normally to the solution  $x^0$  if and only if  $\sigma > 0$ .

An l.r.p. is called **quasigradient** if  $\inf |\cos \theta_k| > 0$ , and is called **strictly relaxation** if  $\inf [q_k(2 - q_k)] > 0$ . For quasigradient and strictly relaxation l.r.p., the formulations of Theorems 1-3 are naturally simplified (see (1)). In particular, the following holds.

**Theorem 4.** If an l.r.p. is simultaneously quasigradient and strictly relaxation, then it converges normally to the solution  $x^0$ .

For illustration let us consider a class of processes of the form

$$x_{n+1} = x_n - \beta_n A_n F(x_n), \quad (7)$$

where  $A_n = A_n^* > 0$  are bounded operators in  $H$ ,  $\beta_n > 0$ . The convergence of these processes was studied in the recent paper of M. N. Yakovlev (4) and in a number of earlier works (see the bibliography in (4)). It is not difficult to show that, under the conditions imposed in (4) on the operators  $A_n$  and the numbers  $\beta_n$ , and with a good localization, the processes (7) are quasigradient and strictly relaxation. Consequently, they converge normally to the solution with a rate guaranteed by estimate (6).

Let us now consider, in the  $n$ -dimensional Euclidean space  $H_n$ , coordinate-relaxation processes defined by formula (3), where  $y_k = e_{i_k}$  ( $\{e_s\}_1^n$  is an orthonormal basis in  $H_n$ ). S. Shekhter showed (5) that if, in the controlling sequence  $\{i_k\}_0^\infty$  of a well-localized strictly relaxation process, each index from  $\Omega^n = \{1, 2, \dots, n\}$  occurs arbitrarily far along (condition  $S$ ), then  $\{x_k\}_0^\infty$  converges to  $x^0$ . Yu. I. Lyubich, in (6), under condition  $S$  obtained an estimate of the rate of convergence for the case of a linear equation. We transfer this result below to the nonlinear equation (1).

Notation:

$$\lambda(N) = \min\{l : \Omega^n \subset \{i_k\}_N^{N+l-1}\};$$

$$N_{p+1} = N_p + \lambda(N_p) \quad (p = 0, 1, 2, \dots; \quad N_0 = 0);$$

$$\nu(k) = \min\{p : N_{p+1} > k\}; \quad \varepsilon_p = \min q_k(2 - q_k) \quad (N_p \leq k < N_{p+1}).$$

\* In deriving these estimates, inequalities between successive derivatives are used, analogous to the well-known Landau-Hadamard inequality (see, for example, (3)).

**Theorem 5.** For coordinate l.r.p., under condition  $S$ , the inequality holds

$$\Delta_k \leq \prod_{p=0}^{\nu(k)-1} (1 - \omega \varepsilon_p) \quad (k = 0, 1, 2, \dots),$$

where the coefficient  $\omega$  ( $0 < \omega \leq 1$ ) depends only on the dimension  $n$  and on the condition measure  $h$ , and moreover  $\omega(n, h)$  is a decreasing function of  $h$ .

From Theorem 5 there follow, in a purely formal way, consequences analogous to those formulated in (6), in particular the above-mentioned result of S. Schechter.

The scheme considered does not formally cover relaxation processes for the Rayleigh functional  $\varphi(x) = (Ax, x)/(x, x)$ . However, it can be shown that here too theorems analogous to our Theorems 1-5 are valid.

Assumptions and notation:  $A = A^* > 0$  is a bounded operator in  $H$ ;  $\lambda_1 = \inf_H \varphi(x)$  is an isolated point of the spectrum,  $U$  is the corresponding eigenspace;

$G = H \ominus U$ ;  $\lambda_2 = \inf_G \varphi(g)$ ;  $x^0 \in U$ ,  $\|x^0\| = 1$ ;  $\{x_k\}_0^\infty$  is an r.p.,  $\|x_k\| = 1$ ,  $\varphi(x^0) < (\lambda_1 + \lambda_2)/2$  (the last condition ensures good localization).

Using the representation  $x = \beta u + \gamma g$  ( $x \in H$ ,  $u \in U$ ,  $g \in G$ ,  $\|u\| = \|g\| = 1$ ), introduce the operator  $P$ :  $Px = |\beta|x^0 + \gamma g$ . Further set  $Px_{k+1} - Px_k = -\alpha_k y_k$ , where  $\alpha_k = \|Px_{k+1} - Px_k\|$ . Let us now note that

$$\text{grad } \varphi(x) = \frac{2}{(x, x)} R(x)x \quad (R(x) = A - \varphi(x)E),$$

and put  $R_k = R(x_k)$ . It can be shown that the equality

$$\Delta \varphi(x_k) = -q_k(2 - q_k) \cos^2 \theta_k \Delta_k h_k, \quad (8)$$

holds, where  $q_k = \alpha_k(R_{ky}k, y_k)/(R_{kPx}k, y_k)$ ;  $\cos \theta_k = (\text{grad } \varphi(x_k), y_k)/\|\text{grad } \varphi(x_k)\|$ ;  $\Delta_k = \varphi(x_k) - \lambda_1$ ;  $h_k = \|R_{kPx}k\|^2[(R_{ky}k, y_k)\Delta_k]^{-1}$ . The inequalities

$$h^{-1}(x_k, A) \leq h_k \leq h(x_k, A) \quad (k = 1, 2, \dots), \quad (9)$$

hold, where  $h(x, A)$  is a certain condition measure, and as  $\varphi(x) \searrow \lambda_1$ ,  $h(x, A) \searrow (\|A\| - \lambda_1)(\lambda_2 - \lambda_1)^{-1}$ , the condition measure of the operator  $R(x^0)|_G$ . Relations (8)-(9) make it possible to obtain, for the Rayleigh functional, Theorems 1-5 in the corresponding formulation.

In conclusion, the author takes this opportunity to express gratitude to Yu. I. Lyubich for posing the problem and for his constant attention.

Physical-Technical Institute of Low Temperatures  
Academy of Sciences of the Ukrainian SSR

Received  
10 XII 1966

## CITED LITERATURE

1. Yu. I. Lyubich, DAN, 161, No. 6, 1274 (1965).
2. M. M. Vainberg, *Variational Methods for the Study of Nonlinear Operators*, Moscow, 1956.
3. G. H. Hardy, J. E. Littlewood, G. Polya, *Inequalities*, IL, 1948.
4. M. N. Yakovlev, Tr. Matem. Inst. im. V. A. Steklova AN SSSR, 84, 8 (1965).
5. S. Schechter, Trans. Am. Math. Soc., 104, No. 1, 179 (1962).
6. Yu. I. Lyubich, DAN, 173, No. 1, 37 (1967).

*Note: Figure translations are in progress. See original paper for figures.*

*Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.*