

The second boundary value problem for the differential equation of elastic equilibrium of a slanting cylindrical shell

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Abstract

The Lauricella method is applied to reduce the boundary value problem of the equilibrium of a part of a cylindrical shell, bounded by a simply connected smooth contour, to integral equations. The components of the displacement vector and the rotation angles of the shell normal are specified on the contour. The kernels of the resulting Fredholm equations of the second kind contain Hankel functions of zero index. In the case of a slightly curved shell, the solvability of these equations follows from the contraction mapping principle and the solvability of the analogous Lauricella-Sherman equations, to which the second boundary value problem for a plate is reduced. Bibliography 6.

Full Text

Preamble

In 1967, the study of boundary value problems for differential equations remained a central focus in mechanics, particularly in the theory of elasticity and shells. Following the foundational work in [?], we consider the complex-valued function $W(x, y)$ satisfying the equation:

$$\Delta\Delta W(x, y) - ik^2\Delta W(x, y) = 0$$

where the boundary conditions are defined by functions $a(s)$ and $b(s)$ along the contour L . Specifically, the equilibrium of the system requires that:

$$\int_L [a(s)dx + b(s)dy] = 0$$

The function $W(x, y)$ is related to the physical displacement $w(x, y)$ and the stress function $\phi(x, y)$ of a shallow spherical shell. According to the theory

established in [?], we define:

$$W(x, y) = w(x, y) + iC\phi(x, y)$$

where the constant C depends on the shell geometry and material properties:

$$C = \frac{\sqrt{12(1-\nu^2)}}{Eh^2}, \quad k^2 = \frac{\sqrt{12(1-\nu^2)}}{Rh}$$

Here, R is the radius of curvature, h is the thickness, E is Young' s modulus, and ν is Poisson' s ratio.

II. Stress and Displacement Relations

The internal forces and moments within the shell can be expressed through the derivatives of $W(x, y)$. Following the formulations in [?, ?], the stress components $\sigma_x, \sigma_y, \tau_{xy}$ and the displacements u, v are integrated across the thickness z . We define the auxiliary functions u_0, v_0 and the rotation χ_0 such that the boundary conditions on the contour L can be represented in terms of the complex potential.

The relationship between the physical components and the complex function W is given by:

$$\begin{aligned} 2\mu h C u_1(x, y) &= \text{Im} \left[\frac{1+\nu}{2\mu h C} \int (\dots) dx - ik^2 \int W dy \right] \\ 2\mu h C u_2(x, y) &= \text{Im} [\dots] \end{aligned}$$

where μ is the shear modulus. By introducing the constants A, B_1, B_2 , we account for the rigid-body displacements of the shell. The boundary conditions on the arc length s ($0 < s < l$) for the generalized displacement and force functions $a(s)$ and $b(s)$ are then derived.

III. Integral Representations and Kernels

To solve the governing equation (1), we utilize fundamental solutions and potential theory. Let (ξ, η) be a point in the domain D . The fundamental solution involves terms of the form $\ln(r)$ and modified Bessel functions. Specifically, we define the kernels N_1 and N_2 as:

$$\begin{aligned} N_1(u, v, \chi) &= -\frac{\partial}{\partial n} \ln r + \dots \\ N_2(u, v, \chi) &= \dots \end{aligned}$$

As shown in [?], the asymptotic behavior of these kernels near the singularity $r \rightarrow 0$ involves terms like $\ln(i\alpha r)\{1 + O(\alpha^2 r^2)\}$.

We represent the solution $W(x, y)$ as an integral over the boundary L using density functions $p_0(s)$ and $q_0(s)$:

$$W(x, y) = \int_L [u^* p_0(s) - v^* q_0(s)] ds$$

where u^* and v^* are derived from the fundamental solutions of the operator $\Delta(\Delta - ik^2)$. The densities $p_0(s)$ and $q_0(s)$ are related to the physical boundary values through a system of singular integral equations.

The geometric kernels G_1, G_2, G_3, G_4 are introduced to simplify the expressions for the stresses and displacements. These kernels depend on the relative coordinates and the parameter $\alpha = \sqrt{ik}$. For instance:

$$G_1(\rho, t) = - \int \frac{\rho^2 \sqrt{1 - \rho^2} \cos \psi}{\dots} d\rho$$

These functions allow us to construct the influence coefficients Q_{nm} and T_{nm} that form the matrix of the integral equation system.

IV. System of Integral Equations

The boundary value problem is ultimately reduced to a system of four integral equations for the unknown densities $p_\nu(s)$ ($\nu = 1, 2, 3, 4$):

$$x_n p_n(s_0) + \int_L \sum_{m=1}^4 K_{nm}(s, s_0) p_m(s) ds = f_n(s_0)$$

where x_n are constants (e.g., $x_1 = x_3 = 1 + \nu$). The right-hand side functions $f_n(s_0)$ are determined by the prescribed boundary displacements and forces.

The kernels K_{nm} contain both logarithmic singularities and regular parts. The existence and uniqueness of the solution $P(s)$ are analyzed using the Fredholm alternative and the properties of the operator in the space $C(L)$. For a shell with a fixed boundary, the system can be written in operator form as:

$$(I + \lambda^2 B)P = F$$

where B is a compact operator. As demonstrated in [?, ?], for sufficiently small λ , the solution can be obtained via successive approximations or by direct inversion of the singular part. The condition for the existence of a single-valued displacement field is ensured by the integral constraint:

$$\int_L [p_1(s)x'(s) + p_4(s)y'(s)] ds = 0$$

which corresponds to the global equilibrium of the shell segment.

References

1. Bakushinsky, S. M., & Isakov, M. A. (1965). *Journal of Computational Mathematics and Mathematical Physics*, 5(2), 219–226.
2. Timoshenko, S. P., & Woinowsky-Krieger, S. (1963). *Theory of Plates and Shells*.

3. Muskhelishvili, N. I. (1935). *Some Basic Problems of the Mathematical Theory of Elasticity*.
4. Kantorovich, L. V., & Akilov, G. P. (1959). *Functional Analysis in Normed Spaces*.
5. Smirnov, V. I. (1957). *A Course of Higher Mathematics*, Vol. IV.

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