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# SYMMETRIZABLE SPACES

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**Abstract**

**Full Text**

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*MATHEMATICS*

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## SYMMETRIZABLE SPACES AND FINAL COMPACTNESS

*(Presented by Academician P. S. Aleksandrov, 4 X 1966)*

All the proofs in this article are based on Theorem 1; its formulation is based on the notion, recently proposed by A. V. Arhangel'skii, of a weakly discrete system of sets. The proof of Theorem 1 rests on a technique developed by Bing for the case of spaces with a refining sequence of covers and subsequently applied by MacAuley in the study of strongly symmetrizable spaces.

The proof of Theorem 2 is based on the main idea of the proof of the theorem on the metrizable of a symmetrizable bicomactum, due to A. V. Arhangel'skii and V. Stoyanovsky (see <sup>(1)</sup>). Theorem 2 may be regarded as a partial answer to Michael's question: is every closed set in a symmetrizable space a set of type  $G_\delta$ .

**Definition 1** (A. Arhangel'skii). A system  $\{M_\alpha\}_{\alpha \in A}$  of subsets of a topological space  $X$  is called **weakly discrete** if from  $x_\alpha \in M_\alpha$  it follows that the system of points  $\{x_\alpha\}_{\alpha \in A}$  is discrete in  $X$ .

**Theorem 1.** *In every open cover of a symmetrizable space one can inscribe a  $\sigma$ -weakly discrete cover.*

**Proof.** Let  $X$  be a space with symmetrix  $d$ . We shall call a system of subsets  $\{M_\alpha\}_{\alpha \in A}$  **discrete with respect to the symmetrix  $d$**  if for every  $x \in X$  there exists  $\varepsilon = \varepsilon(x) > 0$  such that  $O_\varepsilon x$  meets no more than one element of the system  $\{M_\alpha\}_{\alpha \in A}$ .

**Lemma.** *If the system  $\{M_\alpha\}_{\alpha \in A}$  is discrete with respect to the symmetrix  $d$  and, for every  $\alpha \in A$ ,  $M_\alpha$  is closed in  $X$ , then the system  $\{M_\alpha\}_{\alpha \in A}$  is discrete in  $X$ .*

Indeed, if for some  $A' \subset A$  the set  $\bigcup_{\alpha \in A'} M_\alpha$  is not closed in  $X$ , then there exists a point

$$x \in X \setminus \bigcup_{\alpha \in A'} M_\alpha$$

such that

$$d\left(x, \bigcup_{\alpha \in A'} M_\alpha\right) = 0.$$

But the system  $\{M_\alpha\}_{\alpha \in A}$  is discrete with respect to the symmetrix  $d$ ; consequently,

$$d(x, M_{\alpha_0}) = 0$$

for some  $\alpha_0 \in A'$ . Hence  $x \in M_{\alpha_0}$ , and this contradicts the condition

$$x \in X \setminus \bigcup_{\alpha \in A'} M_\alpha.$$

Let now  $\omega = \{U_\alpha\}_{\alpha \in A}$  be an open cover of the space  $X$ , with  $A$  a well-ordered set. For each natural  $n$  and each  $\alpha \in A$ , put

$$M_\alpha^n = \mathcal{E}\{x \in X, x \in U_\alpha \setminus \bigcup_{\beta < \alpha} U_\beta, O_{1/n}x \subset U_\alpha\}.$$

We shall show that, for each fixed  $n$ , the system  $\{M_\alpha^n\}_{\alpha \in A}$  is discrete with respect to the symmetrix  $d$ . Indeed, let  $x \in X$ . Then, for some  $\alpha \in A$ ,

$$x \in U_\alpha \setminus \bigcup_{\beta < \alpha} U_\beta$$

and, for some  $m \geq n$ ,

$$O_{1/m}x \subset U_\alpha$$

(whence it is clear that  $\{M_\alpha^n\}_{\alpha \in A, n=1,2,\dots}$  is a cover). Suppose that for some  $\beta$ ,

$$O_{1/m}x \cap M_\beta^n \neq \Lambda.$$

This is impossible for  $\beta > \alpha$ , since in that case

$$M_\beta^n \subset U_\beta \setminus U_\alpha,$$

whereas

$$O_{1/m}x \subset U_\alpha.$$

But  $O_{1/m}x \cap M_\beta^n \neq \Lambda$  also cannot occur for  $\beta < \alpha$ , since then, from

$$y \in O_{1/m}x \cap M_\beta^n,$$

it follows that

$$x \in O_{1/n}y \subset O_{1/m}y \subset U_\beta,$$

which contradicts the relation

$$x \in U_\alpha \setminus \bigcup_{\gamma < \alpha} U_\gamma \subset U_\alpha \setminus U_\beta.$$

To complete the proof of Theorem 1 it remains only to apply the lemma.

**Theorem 2.** *A finally compact symmetrizable space is hereditarily finally compact.*

**Proof.** Let  $X$  be a Hausdorff finally compact space with a symmetric  $d$ , and let  $F$  be a closed subspace of the space  $X$ . Then  $X' = X \setminus F$  is a symmetrizable space, as an open subspace of a symmetrizable space. Let  $\{U_\alpha\}_{\alpha \in A}$  be an open cover of the space, where  $A$  is a well-ordered set. Put again

$$M_\alpha^n = \mathcal{E}\{x \in X', x \in \overline{U}_\alpha \setminus \bigcup_{\beta < \alpha} U_\beta, O_{1/n}x \subset U_\alpha\}$$

and suppose that for some  $n$  the set of nonempty elements of the system  $\{M_\alpha^n\}_{\alpha \in A}$  is uncountable. Choose from each nonempty element of the system  $\{M_\alpha^n\}_{\alpha \in A}$  a point  $x_\alpha \in M_\alpha^n$ .

Since

$$F = \bigcap_{m=1}^{\infty} O_{1/m}F,$$

there exist  $\varepsilon > 0$  and  $A' \subset A$  such that:

- 1)  $x_\alpha \notin O_\varepsilon F$  for  $\alpha \in A'$ , and 2) the system  $\{x_\alpha\}_{\alpha \in A'}$  is uncountable.

The set  $\mathcal{E}\{x_\alpha : \alpha \in A'\}$  is not closed in  $X$ , because, as we saw in the proof of Theorem 1, it is discrete in the space  $X'$ , and it has limit points by virtue of the final compactness of the space  $X$  and the large cardinality of the set  $\mathcal{E}\{x_\alpha : \alpha \in A'\}$ . Consequently, there exists a point  $x_0 \in X \setminus \mathcal{E}\{x_\alpha : \alpha \in A'\}$  such that  $d(x_0, \mathcal{E}\{x_\alpha : \alpha \in A'\}) = 0$ .

From the last equality and the definition of the set  $A'$  it follows that  $x_0 \in X'$ , but this contradicts the discreteness of the system  $\{x_\alpha\}_{\alpha \in A'}$  in  $X$ . The contradiction completes the proof of Theorem 2, since it shows that from the cover  $\{M_\alpha^n\}_{\alpha \in A, n=1,2,\dots}$  and consequently also from the cover  $\{U_\alpha\}_{\alpha \in A}$ , one can choose a countable subcover.

**Theorem 3.** *A symmetrizable space is finally compact if and only if every discrete system of points in it is countable.*

**Proof** follows directly from Theorem 1, the lemma, and the known fact that in any finally compact space every discrete system is countable.

**Corollary 1.** *A Hausdorff symmetrizable hereditarily separable space is hereditarily finally compact\*.*

**Corollary 2.** *A Hausdorff countably compact symmetrizable space is compact\*\*.*

**Proof.** If  $X$  is a countably compact space, then in  $X$  every discrete system of points is finite. Hence, if  $X$  is a symmetrizable space, then by Theorem 3 it follows that  $X$  is finally compact and, consequently, bcompact. By the main

theorem of Arhangel'skii and Stoyanovskij formulated at the beginning <sup>(1)</sup>, \*\*\* a symmetrizable bicomactum is compact.

**Theorem 4.** *A symmetrizable  $M$ -space (in the sense of Morita <sup>(6)</sup>) is metrizable.*

**Proof.** By Morita's theorem (<sup>(6)</sup>, Theorem 5.3), every  $M$ -space admits a closed mapping onto a metric space under which the preimage of each point is a countably compact space. Hence, from Corollary 2 we obtain that every symmetrizable  $M$ -space is a feathered paracompactum (see <sup>(3)</sup>), and an arbitrary feathered paracompactum with a symmetric is metrizable, by a theorem of A. V. Arhangel'skii.

\* If the word symmetrizable is replaced by strongly symmetrizable, one obtains a statement proved by Sider <sup>(7)</sup>. His arguments do not carry over to the case of symmetrizable spaces.

\*\* If the word symmetrizable is replaced by strongly symmetrizable, one obtains a statement proved by Nemytskii <sup>(5)</sup>. His arguments do not carry over to the case of symmetrizable spaces.

\*\*\*) The key role in the proof of this theorem is played by the special case of Theorem 2 concerning bicomacta. Then a known theorem of Nemytskii is applied <sup>(5)</sup>.

**Definition 2** (A. Arhangel'skii <sup>(2)</sup>). A symmetric  $d$  of a symmetrizable space  $X$  is called a **symmetric satisfying the weak Cauchy condition\*** if, from  $A \subset X$  and  $[A] \neq A$ , it follows that for every  $\varepsilon > 0$  there are distinct points  $x$  and  $y$  in  $A$  for which  $d(x, y) < \varepsilon$ .

In Arhangel'skii's paper <sup>(2)</sup> the following assertion is proved: a factor, weakly uniformly\*\* image of a metric space is symmetrizable with a symmetric satisfying the weak Cauchy condition.

Using the ideas of <sup>(2)</sup>, we prove the following theorem.

**Theorem 5.** *If the Hausdorff symmetric  $d$  of a finally compact symmetrizable space  $X$  satisfies the weak Cauchy condition, then for every subspace  $X'$  of the space  $X$  and every  $\varepsilon > 0$ , from the cover  $\{O_\varepsilon x\}_{x \in X'}$  of the space  $X'$  one can choose a countable cover of the space  $X'$ .*

**Proof.** Starting with an arbitrary point  $x_0 \in X'$ , we construct a system of points  $\{x_\alpha \in X'\}_{\alpha \in A}$  such that: 1)  $A$  is well ordered; 2)

$$X' = \bigcup_{\alpha \in A} O_\varepsilon x_\alpha;$$

3)

$$x_\alpha \in X' \setminus \bigcup_{\beta < \alpha} O_\varepsilon x_\beta.$$

The last condition can obviously be replaced by 3') for  $\alpha \neq \beta$ ,  $d(x_\alpha, x_\beta) \geq \varepsilon$ . The system of points  $\{x_\alpha\}_{\alpha \in A}$  so constructed turns out to be discrete in  $X$ . Indeed, if we suppose that for some subset  $A'$  of the set  $A$  the set  $\mathcal{E}\{x_\alpha : \alpha \in A'\}$  is not closed in  $X$ , then we would obtain that for some  $\alpha \neq \beta$ ,  $d(x_\alpha, x_\beta) < \varepsilon$ , which contradicts condition 3'). Consequently, the system of points  $\{x_\alpha\}_{\alpha \in A}$  is countable, as is every discrete system of points in a finally compact space. The theorem is proved.

**Corollary 3.** *A symmetrizable finally compact space, some symmetric of which satisfies the weak Cauchy condition, is hereditarily separable.*

**Proof.** Let a symmetric  $d$  of the finally compact symmetrizable space  $X$  satisfy the weak Cauchy condition, and let  $X'$  be an arbitrary subspace of the space  $X$ . By Theorem 5, for every natural  $n$  there exists a sequence of points  $\{x_m^n\}_{m=1}^\infty$  such that: 1)  $x_m^n \in X'$  for every  $m$ , and 2)

$$X' = \bigcup_{m=1}^\infty O_{1/n} x_m^n.$$

It is easy to verify that the countable set  $\mathcal{E}\{x_m^n, n, m = 1, 2, \dots\}$  is everywhere dense in  $X'$ .

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## REFERENCES

1. A. V. Arhangel'skii, UMN, 21, no. 4 (130), 133 (1966).
2. A. V. Arhangel'skii, DAN, 164, no. 2 (1965).
3. A. V. Arhangel'skii, Matem. sborn., 67 (109), no. 1 (1965).
4. P. S. Aleksandrov, V. V. Nemytskii, Matem. sborn., 3, 3, 665 (1938).
5. V. Niemytzki, Math. Ann., 101, no. 5, 666 (1931).
6. K. Morita, Proc. Japan Acad., 39, 150 (1963).

7. J. G. Geder, Pacific J. Math., 11, no. 1 (1961).

\* A symmetric  $d$  is called a **Cauchy symmetric** (see (4)) if, from

$$\lim_{n \rightarrow \infty} d(x_n, x) = 0,$$

it follows that

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0.$$

\*\* A mapping  $f : X \rightarrow Y$  of a symmetrizable space  $X$  onto a topological space  $Y$  is called **weakly uniformly continuous** if for any point  $y \in Y$  and its neighborhood  $U$ ,

$$\rho(f^{-1}y, X \setminus f^{-1}U) > 0,$$

where  $\rho$  is a fixed symmetric of the space  $X$ .

*Note: Figure translations are in progress. See original paper for figures.*

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