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Abstract

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SOLUTION OF BOUNDARY-VALUE PROBLEMS FOR NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS OF SECOND ORDER BY THE STABILIZATION METHOD

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Some problems in nonlinear field theory, statistical nuclear theory, and beam extraction from an accelerator lead to the consideration of boundary-value problems for nonlinear differential equations of second order. Along with qualitative investigations of these problems, there arises the need to construct algorithms for numerical solution. Among known methods we note, for example, the iterative finite-difference method ([1]). The proof of convergence of the method is based on the principle of contraction mappings, which imposes rather severe restrictions on the nonlinear part of the equation. Another approach to the problem posed consists in applying to the solution of boundary-value problems for differential equations of second order the general Newton method for solving functional equations, proposed by L. V. Kantorovich. This method, developed by N. N. Glinskaya and I. P. Mysovskikh ([2]), is likewise not free of the shortcomings indicated for the preceding method. The "stabilization" method for solving stationary problems has become widely used. The essence of this method, as applied to the solution of boundary-value problems, is as follows.

Let a nonlinear differential equation be given

$$L[y] = y'' + f(x, y) = 0 \tag{1}$$

with boundary-value problem

$$y(0) = y(1) = 0. \tag{2}$$

Introducing a continuous time parameter t, we consider the nonstationary problem for the function $u(x, t)$:

$$\partial u(x,t)/\partial t = \partial^2 u(x,t)/\partial x^2 + f(x, u(x,t))$$

with boundary condition

$$u(0,t) = u(1,t) = 0$$

and with some initial condition

$$u(x,0) = \varphi(x), \qquad \varphi(0) = \varphi(1) = 0.$$

The solution "stabilizes" if, as $t \to \infty$, $\partial u(x,t)/\partial t \to 0$. In this case the solution of the nonstationary problem converges to the solution of problem (1)–(2). In the case of stabilization one may apply various mesh methods of solution, which makes it possible to avoid the difficulties associated with the nonlinearity of the differential equation. Unfortunately, in works devoted to stabilization, for example ([3]), substantial restrictions are likewise imposed on the function $f(x,y)$ and its derivatives with respect to x and y.

In the present paper a certain generalized method of stabilization with respect to a continuous parameter for solving boundary-value problems of type (1)–(2) is set forth. This method does not impose such substantial restrictions on $f(x,y)$ and its derivatives as were assumed in the methods described above. The method is more general than ([2]): the method set forth in ([2]) is obtained as a particular realization of the proposed method.

Introduce a continuous parameter t into problem (1)–(2) in the following way. Consider a function $y(x,t)$, for which we form the differential relation

$$\partial L[y]/\partial t = -L[y]. \tag{3}$$

The expression $\partial L[y]/\partial t$ can be represented as follows:

$$\frac{\partial L[y]}{\partial t} = \frac{\partial}{\partial t}\left[\frac{\partial^2 y(x,t)}{\partial x^2}\right] + f'_y(x, y(x,t))\frac{\partial y(x,t)}{\partial t}.$$

Assuming that $y(x,t)$ has a continuous derivative

$$\frac{\partial}{\partial t}\left[\frac{\partial^2 y(x,t)}{\partial x^2}\right],$$

and denoting $\partial y(x,t)/\partial t$ by $v(x,t)$, we arrive at the following system of partial differential equations, equivalent to relation (3):

$$\partial^2 v(x,t)/\partial x^2 + f'_y(x, y(x,t))v(x,t) = -\left[\partial^2 y(x,t)/\partial x^2 + f(x, y(x,t))\right],$$

$$\partial y(x,t)/\partial t = v(x,t). \tag{4}$$

For the resulting system consider the following problem: in the half-strip $0 \leq x \leq 1,\ 0 \leq t < \infty$, find functions $v(x,t)$ and $y(x,t)$ satisfying the conditions

$$v(0,t) = v(1,t) = 0, \qquad y(x,0) = \varphi(x), \tag{5}$$

where $\varphi(x)$ is some known twice continuously differentiable function satisfying the boundary condition

$$\varphi(0) = \varphi(1) = 0.$$

If the solution of problem (4)–(5) exists, then, as is easy to see from (3), the solution $y(x,t)$ stabilizes to the solution of problem (1)–(2). Indeed, it follows directly from (3) that

$$L[y] = C(x)e^{-t}, \tag{6}$$

where $C(x)$ is some continuous function. Hence, as $t \to \infty$, $L[y] \to 0$, i.e. the solution stabilizes, $y(x,t) \to y(x)$, where $y(x)$ is a function satisfying (1). Fulfillment of the boundary condition (2) is ensured by conditions (5). One may propose, for example, the following implementation of the proposed method. Choose a step of motion with respect to the parameter t, denoting it by τ. Divide the half-strip $0 \leq x \leq 1,\ 0 \leq t < \infty$ by straight lines parallel to the x-axis, $t = t_i,\ t_{i+1} - t_i = \tau$. Replace the second equation of system (4) by some difference analogue, for example

$$[y(x,t_{i+1}) + y(x,t_i)]/\tau = v(x,t_i). \tag{7}$$

On the layer $t = t_i$ solve the linear ordinary differential equation with respect to $v(x,t_i)$

$$v''(x,t_i) + f'_y(x,y(x,t_i))v(x,t_i) = -\left[y''(x,t_i) + f(x,y(x,t_i))\right], \tag{8}$$

where the function $y(x,t_i)$ is regarded as known, with the boundary-value problem

$$v(0,t_i) = v(1,t_i) = 0. \tag{9}$$

The transition to the $(i+1)$-st layer is carried out with the aid of (7). The numerical solution of problem (8)–(9) can be implemented by any known method.

We formulate the main result of the paper in the form of the following assertion.

Theorem. Let the solution of the boundary-value problem (1)–(2) exist and, in the case of nonuniqueness, be "localized," i.e. let it be possible to construct twice continuously differentiable functions $z(x)$ and $Z(x)$, vanishing at $x = 0$ and $x = 1$, such that in the region

$$z(x) \leq y \leq Z(x) \tag{10}$$

there is only one solution of problem (1)–(2). Suppose that $f(x, y)$ has continuous derivatives with respect to x, y up to the second order inclusive. Suppose further that the following conditions are satisfied:

1) $|\varphi''(x) + f(x, \varphi(x))| \leqslant \varepsilon$, where ε is a sufficiently small positive number;
2) the boundary-value problem

$$v'' + f'_y(x, y(x))v = -[y'' + f(x, y)], \qquad v(0) = v(1) = 0$$

has a unique solution for any continuously differentiable function $y(x)$ satisfying the condition $z(x) \leqslant y(x) \leqslant Z(x)$.

Then the solution of system (4) with conditions (5) exists in the entire half-strip $0 \leqslant x \leqslant 1,\ 0 \leqslant t < \infty$. The approximate solution of the system obtained from (7)–(9) converges uniformly as $t \to 0$ to the solution of problem (4)–(5). The rate of convergence of the approximations under motion along the parameter is determined by the relation

$$|y(x) - y(x, t_i)| \leqslant \varepsilon e^{-t_i} \Phi(x),$$

where $\Phi(x)$ is some bounded positive function that vanishes at $x = 0$ and $x = 1$.

The proof of the existence of a solution of (4)–(5) is based on an analogue of Euler' s polygonal-line method.

The method described was applied to the special case of a singular second-order differential equation arising in nonlinear field theory and in the statistical theory of the nucleus ([4]), with a boundary-value problem on a bounded interval. A positive solution and solutions with one and two zeros inside the interval were found.

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Note: Figure translations are in progress. See original paper for figures.