MATHEMATICS

A. A. ZINGER, Academician Yu. V. LINNIK

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Abstract

Full Text

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ON POLYNOMIAL STATISTICS OF A NORMAL SAMPLE

1. Let $x = (x_1, x_2, \dots, x_n)$ be a repeated normal sample from the population $N(0, 1)$. Many tests of multivariate analysis are constructed by means of independent, usually polynomial or rational, statistics of such a sample. In view of this, a systematic study of independent statistics of the indicated type is of considerable interest.

Here several theorems will be formulated concerning the independence of polynomial statistics of a normal sample. Let us note that our method of investigation makes it possible to reveal certain connections between mathematical statistics and algebraic geometry.

It is natural to attempt to classify all pairs of independent polynomial statistics in a sample from a normal population. So far, however, by using the method proposed here, it is possible to give a characterization only of almost all, in a known sense, such pairs of statistics, and also to single out particular systems of independent polynomials.

If $A(x_1, x_2, \dots, x_n)$ and $B(x_1, x_2, \dots, x_n)$ are independent polynomial statistics, then, by a well-known theorem of R. A. Fisher, any orthogonal transformations $x = (x_1, x_2, \dots, x_n)$ again bring the statistics A and B into an independent pair. We shall say that the statistics A and B split if there exists such an orthogonal transformation under which the statistics A and B reduce to functions of different coordinates. Obviously, statistics that split are independent.

In 1954 a conjecture was stated (see ([1])) according to which there do not exist, in a sample from a normal population, independent non-splitting polynomial statistics. The results of the present note confirm this conjecture in some particular cases.

2. Consider the affine space $\mathfrak{M}_{\alpha\beta}$ of all possible homogeneous pairs of polynomials of degrees α and β, respectively. In this space we single out the set of independent pairs of polynomials. Denote it by $\mathfrak{N}$. If $(A, B) \in \mathfrak{N}$, then, obviously, the decorrelation relations hold

$$EA^{rB^s} - EA^{rEB^s} = 0^*, \quad r, s = 0, 1, \dots$$

By Hilbert' s theorem (see ([2],[3])) these relations form an algebraic set $\mathfrak{D}$, which we shall call the set of decorrelation. This set forms an algebraic correspondence in the space of the coefficients of A and B (see ([3])). Obviously, $\mathfrak{N} \subset \mathfrak{D}$. If the conjecture stated above is true, then conversely $\mathfrak{D} \subset \mathfrak{N}$, but as yet the validity of this has not been established.

By well-known theorems of algebraic geometry (see, for example, ([3])), the algebraic set $\mathfrak{D}$, considered over the field of real numbers, can be represented as the union of a finite number of components–irreducible varieties, each of which is connected. In what follows we shall be interested in this decomposition in the components of highest dimension.

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* $E\{\cdot\}$, as usual, denotes mathematical expectation.

It turns out that, under certain conditions, all points of these components correspond to decomposable polynomials. Namely, the following holds:

Theorem 1. *There exists a constant $\gamma_0(n)$ such that, for $a \geqslant \beta\gamma_0(n)$, the components of highest dimension in the decomposition $\mathfrak{D}$ into irreducible varieties belong to $\mathfrak{R}$, i.e., they are decomposable.**

3. The proof of this theorem is based on applying the transfer method in the multidimensional case and on theorems on the intersection of algebraic varieties over the field of complex numbers. A very important role here is played by an auxiliary theorem, which is evidently also of independent interest.

Theorem 2. *If $A, B \in \mathfrak{D}$, then there exists an orthogonal transformation $x' = O_x$ under which*

$$A = A'(x'_1, x'_2, \dots, x'_{n-1}), \qquad B = B'(x'_2, x'_3, \dots, x'_n),$$

and, moreover, as one of the new coordinate axes one may choose an axis passing through a point of the unit sphere at which the statistic B (respectively A) attains its greatest value on this sphere.

This theorem makes it possible to indicate certain classes of polynomial statistics whose decorrelation entails decomposability. These include, first of all, symmetric statistics, whose use in statistics is most natural. The following holds.

Theorem 3. *Let $B(x_1, x_2, \dots, x_m)$, $m \leqslant n$, be a symmetric homogeneous polynomial statistic. If B attains its greatest value on the sphere*

$$x_1^2 + x_2^2 + \dots + x_m^2 = 1$$

at a point $(x_1^0, x_2^0, \dots, x_m^0)$ which does not belong to the plane

$$x_1 + x_2 + \dots + x_m = 0$$

and to the line

$$x_1 = x_2 = \dots = x_m,$$

then any polynomial statistic $A(x_1, x_2, \dots, x_n)$ decorrelated with B is decomposable with B.

The case in which the condition $\sum x_j \neq 0$ is violated is also of independent interest, since, for example, symmetric statistics of the form $B(x_1 - \bar{x}, x_2 - \bar{x}, \dots, x_n - \bar{x})$, where $\bar{x} = \frac{1}{n} \sum x_j$, are widely used in constructing tests with the location parameter eliminated. Sample moments, widely used in statistics, have this form in particular. Here the following holds.

Theorem 4. *A polynomial statistic $A(x_1, x_2, \dots, x_n)$, decorrelated with a homogeneous polynomial symmetric statistic*

$$B = B(x_1 - \bar{x}, x_2 - \bar{x}, \dots, x_n - \bar{x}),$$

depends only on $\bar{x}$.

4. The possibilities of our method of investigation are not limited to the case of homogeneous statistics. It can also be applied to some nonhomogeneous, rational, and holomorphic statistics of a normal sample. We note that, in general, the theorem formulated above is not true for rational statistics.

In conclusion, let us make a remark about samples from non-Gaussian populations and their polynomial statistics. If the random variable y can be obtained from a normal variable x by means of a "sufficiently good" transformation $y = G(x)$, then in this case too our method is applicable. In doing so it also yields some results on the problem of characterizing distributions by properties of statistics.

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3. B. L. Van der Waerden, *Einführung in die algebraische Geometrie*, Berlin, 1939.

* It follows from this that, in an understandable sense, almost all such statistics are decomposable.