ON LINEAR DIFFERENTIAL GAMES. 1

MATHEMATICS

1967

SovietRxiv

Abstract

Full Text

UDC 517.91

MATHEMATICS

Academician L. PONTRYAGIN

ON LINEAR DIFFERENTIAL GAMES. 1

In the note (2) some results on linear differential games are set forth, derived from the complicated nonlinear theory (1). After my report on January 30 of this year at the conference in Los Angeles, Professor E. Polak (Berkeley) told me that the linear problem is probably solved much more simply directly. Having considered this remark, I came to the conviction that there is a considerably more general result than the one I reported in Los Angeles, which, however, had already been sketched in (2). This more general result I formulated in my lectures in Berkeley, Providence, and Montreal at the beginning of February of this year. In the present note I set forth this result in the very formulation in which it was presented by me in the aforementioned lectures, and give its proof.

Let R be a Euclidean vector space of dimension n; let M be its vector subspace, and let L be the vector subspace of the space R that is the orthogonal complement to the subspace M. We denote the dimension of the space L by ν. Further, let P and Q be two manifolds homeomorphic to the sphere of dimension $\nu - 1$.

In the space R consider the linear differential game (1), ending on the manifold M and described by the differential equation

$$dz/dt = Cz + U(u) - V(v). \tag{1}$$

Here z is a vector of the space R; C is a constant square matrix of dimension n; $u \in P$ and $v \in Q$ are control parameters; U and V are continuous vector functions of these parameters. The parameter u corresponds to the pursuing object, and the parameter v to the evading object. We shall study this differential game (1) under the assumption that the following conditions A and B are satisfied:

A. By π we denote the operation of orthogonal projection from the space R onto the subspace L, and let τ be an arbitrary positive number. We shall assume that the function

$$\pi e^{\tau C} U(u) \tag{2}$$

gives a homeomorphic mapping of the manifold P onto some convex hypersurface in the space L; the convex body bounded in L by this hypersurface will be denoted by $\hat{u}(\tau)$. We shall regard the boundary of the body $\hat{u}(\tau)$ as belonging to this body. In exactly the same way we shall assume that the function

$$\pi e^{\tau C} V(v) \tag{3}$$

gives a homeomorphic mapping of the manifold Q onto some convex hypersurface of the space L. The convex body bounded by this surface will be denoted by $\hat{v}(\tau)$. We shall also regard this body as closed.

B. We shall assume that for any $\tau > 0$ the convex body $\hat{v}(\tau)$ can be moved by a translation (parallel displacement) into the interior of the convex body $\hat{u}(\tau)$.

Here and below, by a convex body we shall mean a convex closed bounded v-dimensional subset of the space L. For the formulation and proof of the result, we introduce some operations on convex bodies.

B. Let A and B be two convex bodies, and let α and β be two real numbers. The set of all vectors of the form

$$z = \alpha x + \beta y, \quad \text{where } x \in A,\ y \in B, \tag{4}$$

is obviously a convex body. We shall denote it by

$$\alpha A + \beta B. \tag{5}$$

It is obvious that if the vector z belongs to the boundary of the body (5), then the vectors x and y (see (4)) belong to the boundaries of the bodies A and B. If $A(\tau)$ is a convex body continuously depending on the real parameter τ on the interval $\tau_1 \leq \tau \leq \tau_2$, then, using the summation operation (5) and passage to the limit, one can define the operation of integration:

$$\int_{\tau_1}^{\tau_2} A(r)\, dr, \tag{6}$$

where its result is the convex body (6). It is obvious that the body (6) consists of those and only those vectors z which can be written in the form

$$z = \int_{\tau_1}^{\tau_2} x(r)\, dr, \quad \text{where } x(r) \in A(r), \tag{7}$$

where $x(\tau)$ is a function of a suitable class. If z belongs to the boundary of the body (6), then almost every point $x(\tau)$ belongs to the boundary of the body $A(\tau)$, so that, without changing the integral (7), one may assume that every point $x(\tau)$ belongs to the boundary of the body $A(\tau)$.

G. Let A and B be two convex bodies such that B can be translated into the interior of A. It is obvious that the set of all vectors x satisfying the condition

$$x + B \subset A \tag{8}$$

constitutes a convex body. We shall denote it by $A * B$. This operation of "subtraction" is entirely different from the usual one (see B).

Theorem. Let z_0 be an arbitrary point of the space R not belonging to M, and let $\tau > 0$. Put

$$\eta(\tau) = \pi e^{\tau C} z_0; \tag{9}$$

$$\hat{w}(\tau) = \hat{u}(\tau) - \hat{v}(\tau); \quad \hat{W}(\tau) = \int_0^\tau \hat{w}(r)\, dr \tag{10}$$

(see A, B, C, G).

For small values of τ, the point $-\eta(\tau)$, obviously, does not belong to the convex body $\hat{W}(\tau)$. If for some values of τ the inclusion

$$-\eta(\tau) \in \hat{W}(\tau) \tag{11}$$

holds, and τ_0 is the minimal value of τ for which the inclusion (11) holds, then, starting from the state z_0, the game can be completed, moreover in a time not exceeding the number

$$T(z_0) = \tau_0. \tag{12}$$

In the proof of this theorem the control u will be constructed with allowance for the control v, so as, if possible, to reduce the time of termination of the game. In constructing the control $u(t_1)$ at the time mo-

we shall use the value $z(t_1)$ at the same moment of time and the control $v(t)$ on the interval $t_1 \leq t \leq t_1 + \varepsilon$, where ε is an arbitrarily small positive number. In the pursuit problem such a formulation of the question is quite admissible; it arises in the case when the pursuing object is chasing not the escaping object itself, but the place where the escaping object was ε seconds earlier. To solve the problem in the usual formulation one should pass to the limit as $\varepsilon \to 0$.

Proof. We shall assume that the control $v(t)$ is given on the interval $0 \leq t \leq \varepsilon$, and let $u(t)$ be, for the time being, an arbitrary control given on the same interval. Substituting these controls into equation (1), we find its solution $z(t)$ on the interval $0 \leq t \leq \varepsilon$ under the initial condition $z(0) = z_0$. The number $T(z(\varepsilon))$ (see (12)) is a functional of the function $u(t)$. Below we shall choose the function $u(t)$ in such a way that the number $T(z(\varepsilon))$ attains its minimal value, and we shall prove that

$$T(z_0) - T(z(\varepsilon)) \geq \varepsilon. \tag{13}$$

It is easy to see that for arbitrary $\tau \geq \varepsilon$ the inclusion

$$\hat{w}(r) \subset \hat{u}(r) - \pi e^{rC} V(v(\tau - r)), \quad \text{where } \tau - \varepsilon \leq r \leq \tau. \tag{14}$$

holds. Integrating this inclusion with respect to r over the limits $\tau - \varepsilon \leq r \leq \tau$ and adding to the resulting relation the convex body $\hat{W}(\tau - \varepsilon)$, we obtain

$$\hat{W}(\tau) \subset \int_{\tau-\varepsilon}^{\tau} [\hat{u}(r) - \pi e^{rC} V(v(\tau - r))] \, dr + \hat{W}(\tau - \varepsilon). \tag{15}$$

By the hypothesis, for $\tau = \tau_0$ the left-hand side of this inclusion contains the point $-\eta(\tau_0)$; consequently, the right-hand side also contains it. Let

$$\tau_1 \leq \tau_0 \tag{16}$$

be the minimal value of τ for which the point $-\eta(\tau)$ is contained in the right-hand side of inclusion (15). Then there exists a control $u(t)$ (see B), $0 \leq t \leq \varepsilon$, such that the point

$$-\left\{ \pi e^{\tau_1 C} z_0 + \int_{\tau_1-\varepsilon}^{\tau_1} \pi e^{rC} [U(u(\tau_1 - r)) - V(v(\tau_1 - r))] \, dr \right\} = -\pi e^{(\tau_1 - \varepsilon)C} z_1, \tag{17}$$

where

$$z_1 = e^{\varepsilon C} z_0 + \int_0^{\varepsilon} e^{sC} [U(u(\varepsilon - s)) - V(v(\varepsilon - s))] \, ds, \tag{18}$$

belongs to the body $\hat{W}(\tau_1 - \varepsilon)$, and this means that

$$T(z_1) \leq \tau_1 - \varepsilon \leq \tau_0 - \varepsilon. \tag{19}$$

Since $z_1 = z(\varepsilon)$ (see (18)), inequality (13) is proved.

Remark. There is no need to assume that Q is a manifold. Let Q be an arbitrary compact set such that, for arbitrary positive τ, the set $\pi e^{\tau C} V(Q)$ can be translated into the interior of the convex body $\hat{u}(\tau)$. Then define the convex body $\hat{w}(\tau)$ as the set of all such vectors x that

$$x + \pi e^{\tau C} V(Q) \subset \hat{u}(\tau).$$

Then the theorem stated above holds, and its proof is preserved in full.

V. A. Steklov Mathematical Institute
Academy of Sciences of the USSR

Received
6 IV 1967

REFERENCES

1. L. S. Pontryagin, *UMN*, **21**, no. 4 (130) (1966).
2. E. F. Mishchenko, L. S. Pontryagin, *DAN*, **174**, no. 1 (1967).

Note: Figure translations are in progress. See original paper for figures.