

The theory of Macdonald integrals. II. Asymptotic expansions

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Abstract

The article is a continuation of the author's work [1] and is devoted to the derivation of two types of asymptotic expansions for Macdonald integrals n of order $M_n(x, y)$. Two expansions of the first type are asymptotic as $|y|\sqrt{1+x^2} \rightarrow \infty$. The first expansion is an expansion in terms of the asymptotic sequence $(H_{m+\alpha}^1(y\sqrt{1+x^2})/(y\sqrt{1+x^2})^m)_{m \in N}$, where $H_\nu^{(1)}(z)$ is the Hankel function of the first kind; α is a real number; N is the set of natural numbers. The second asymptotic expansion of the first type is an expansion in terms of the asymptotic sequence $((y\sqrt{1+x^2})^{-m})_{m \in N}$, which follows from the first one. Both expansions are uniform with respect to x for $|x| \geq \varepsilon > 0$, where ε is an arbitrarily small number. The asymptotic expansion of the second type is valid for $|y| \rightarrow \infty$ and in the case where the parameter $y(\sqrt{1+x^2}-1)$ remains finite as $|y| \rightarrow \infty$, i.e., x must simultaneously tend to zero such that the expression $y(\sqrt{1+x^2}-1)$ remains finite. The presented expansion of the second type is an expansion in terms of the asymptotic sequence $(y^{-m})_{m \in N}$. Bibliography: 4 items.

Full Text

Preamble

In this section, we refine the asymptotic properties of the function $M_n(x, y)$ introduced in [1]. We consider the behavior of the function as $|y|\sqrt{1+x^2} \rightarrow \infty$ and $|y|\sqrt{1+x^2} \rightarrow 0$. Specifically, for $|x| > \epsilon > 0$, we establish estimates for the remainder terms and provide expansions in terms of the parameter y .

§ 1. Asymptotic Definitions and Notation

Let x_0 be a limit point in the domain U . We say that $\phi(x) = O(\psi(x))$ as $x \rightarrow x_0$ if there exists a constant A such that $|\phi(x)| < A|\psi(x)|$ for all x in a neighborhood U_δ of x_0 . Similarly, $\phi(x) = o(\psi(x))$ if $\phi(x)/\psi(x) \rightarrow 0$ as $x \rightarrow x_0$.

An asymptotic series is defined as:

$$f(x) \sim \sum_{k=0}^{\infty} \phi_k(x), \quad x \rightarrow x_0 \quad (1.1)$$

if for every $n \in N$, the following condition holds:

$$f(x) = \sum_{k=0}^n \phi_k(x) + o(\phi_n(x)), \quad x \rightarrow x_0 \quad (1.2)$$

Equivalently, this can be expressed using the O -notation as:

$$f(x) = \sum_{k=0}^n \phi_k(x) + O(\phi_{n+1}(x)) \quad (1.3)$$

In the context of our analysis, we often encounter expansions of the form:

$$f(x) = \phi(x) + \psi(x) \left\{ \sum_{k=0}^n a_k \phi_k(x) + o(\phi_n(x)) \right\} \quad (1.4)$$

which implies the asymptotic equivalence:

$$f(x) \sim \phi(x) + \psi(x) \sum a_k \phi_k(x) \quad (1.5)$$

§ 2. Asymptotic Expansion for Large Arguments

We consider the domain $-\pi < \arg z < 2\pi$ as $|z| \rightarrow \infty$. Let the sequence be defined by $(H_{n+a}(z)/z^n)_{n \in N}$. We assume the property:

$$\frac{H_{n+1+a}(z)}{z^{n+1}} = o\left(\frac{H_{n+a}(z)}{z^n}\right) \quad (2.1)$$

For the function $M_n(x, y)$, as $|y|\sqrt{1+x^2} \rightarrow \infty$, we derive the following representation:

$$M_n(x, y) \sim \frac{H_{n-1/2}^{(1)}(y)}{\sqrt{\pi}(1+x^2)^{n/2}} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(k+1/2)}{2^k \Gamma(1/2)} \frac{|x|^{2k+1}}{(1+x^2)^k} \frac{H_{n-k-1}^{(1)}(y\sqrt{1+x^2})}{(y\sqrt{1+x^2})^{k+1}} \quad (2.2)$$

The remainder term $R_k(x, y)$ for the expansion in (2.2) is given by:

$$R_k(x, y) = \frac{(-1)^{k+1}}{2^{k+1} \Gamma(k+3/2)} \int_1^{\infty} H_{n-k-2}^{(1)}(yu\sqrt{1+x^2})(u^2-1)^{k+3/2} du \quad (2.3)$$

For $x > 0$ and $x < 0$, the behavior of the integral depends on the sign of the argument. Using the properties of Hankel functions $H_n^{(1)}(z)$, we can bound the remainder $|R| < A$ for $|x| > \epsilon > 0$.

By applying the integral representation from \cite[1]:

$$M_n(x, y) = \int_0^{\operatorname{arsh} x} H_n^{(1)}(y \cosh \xi) (\cosh \xi)^{n-1} d\xi \quad (2.5)$$

and substituting the series for the Hankel function, we obtain the coefficients for (2.2). Specifically, using the relation:

$$H_{n-1}^{(1)}(z) = \exp(iz) \sum \dots \quad (2.9)$$

we verify that the k -th term in (2.3) matches the required asymptotic order.

§ 3. Expansion for Small Arguments

As $|y| \rightarrow 0$, we investigate the behavior of $M_n(x, y)$ in the domain $0 < \arg y < \pi$. The expansion takes the form:

$$M_n(x, y) \sim \frac{H_n^{(1)}(y) \operatorname{sign}(x)}{\sqrt{y(\sqrt{1+x^2}-1)}} \int e^{iu^2} du + \dots \quad (3.1)$$

where the coefficients c_n are determined by the recurrence relations:

$$c_n = \frac{(-1)^l (2l+1)!! (2p-1)!! \Gamma(n+m-l)}{(2m+1)!! 2^{2p} (l+1-p)! (m-l)!} \quad (3.2)$$

For $x > 0$, we use the substitution $u = \sqrt{2y \sinh(\xi/2)}$. The integral (3.3) is then evaluated using the properties of the Gamma function and the asymptotic expansion of $H_n^{(1)}(z)$ as $z \rightarrow 0$.

Applying Lemma 2, we find that for $|x| > \epsilon > 0$:

$$R_k(x, y) = O\left(\frac{\exp(iy)}{y^{k+3}}\right) \quad (3.5)$$

The final asymptotic form for $M_n(x, y)$ as $|y|\sqrt{1+x^2} \rightarrow 0$ is:

$$M_n(x, y) \approx \frac{\operatorname{sign}(x)}{\sqrt{\pi}(1+x^2)^{n/2-1}} \sum_{m=0}^k \frac{(n+m-1)!}{m!(n-m-1)!} \frac{1}{(2iy)^m} \quad (3.12)$$

This result is consistent with the general theory of confluent hypergeometric functions and the specific bounds established in \cite[4].

References

1. T. A. A., *Journal of Mathematical Physics*, No. 10, 1967.
2. D. A. A., *Asymptotic Expansions*, 1962.
3. I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 1965.
4. *Higher Transcendental Functions*, Vol. 2, 1966.

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