ON FACTOR REPRESENTATIONS OF TYPE \(\mathrm{II}_1\)

MATHEMATICS

1967

SovietRxiv

Abstract

Full Text

UDC 517.433

MATHEMATICS

V. Ya. GOLODETS

ON FACTOR REPRESENTATIONS OF TYPE II_1

FOR THE CLIFFORD ALGEBRA

(Presented by Academician I. M. Vinogradov on VI 6, 1966)

1. A **representation of the Clifford** (or spinor) **algebra with an infinite number of generators** is a set of linear self-adjoint operators $\{A_k\}_1^\infty$, acting in a separable Hilbert space H and satisfying the relations

$$A_k^2 = 1, \qquad A_{kA}l + A_{lA}k = 0 \quad (k \neq l). \tag{1}$$

If the weakly closed ring M, generated by the operators $\{A_k\}_1^\infty$, is a factor in the sense of Murray and von Neumann, then such a representation will be called a **factor representation**.

Some properties of factor representations were noted in (^{6}). In the present note we give a complete classification of factor representations of type II_1, and also consider the question of decomposing a factor representation of type II_1 into irreducible representations.

2. Let $\{A_k\}_1^\infty$ and $\{B_k\}_1^\infty$ be two factor representations acting in the spaces H_1 and H_2; denote the corresponding factors by M_1 and M_2.

Definition. The factor representations $\{A_k\}_1^\infty$ and $\{B_k\}_1^\infty$ will be called **algebraically isomorphic** if one can establish an algebraic isomorphism φ between the factors M_1 and M_2 in such a way that $\varphi(A_k) = B_k$ $(k = 1, 2, \ldots)$.

It can be shown that there exist algebraically nonisomorphic factor representations of type III and algebraically nonisomorphic representations of type II_∞.

Theorem 1. *All factor representations of the Clifford algebra of type* II_1 *are algebraically isomorphic.*

We outline the proof of the theorem. Introduce in the ring M_i $(i = 1, 2)$ a scalar product by putting

$$\langle A, B \rangle = T_i(AB^*),$$

where T_i is the relative trace for the factor M_i, and $A, B \in M_i$. Then M_i becomes a pre-Hilbert space. We denote by $Q(M_i)$ the completion of M_i with respect to this scalar product. We shall assume that $T_i(I_i) = 1$, where I_i is the identity operator. Consider monomials of the form

$$A_z = A_{s_1} A_{s_2} \cdots A_{s_n},$$

where z denotes a set of indices $s_1 < s_2 < \cdots < s_n$; $n = 1, 2, \ldots$; $s_i = 1, 2, \ldots$ $(i = 1, 2, \ldots, n)$. It is not difficult to verify that

$$T_1(A_z) = 0$$

for all z. Consequently, the monomials of the form A_z and the identity operator I_1 form a complete orthonormal basis in the space $Q(M_1)$. Every element D of $Q(M_1)$ can be expanded in a series with respect to this basis,

$$D \sim \sum_z d_z A_z \qquad \left(\sum_z |d_z|^2 < \infty \right).$$

Now establish a correspondence between $Q(M_1)$ and $Q(M_2)$ by the rule

$$\varphi(I_1) = I_2; \qquad \varphi(A_z) = B_z.$$

It is not difficult to verify that $\varphi(C + D) = \varphi(C) + \varphi(D)$ and $\varphi(C^*) = \varphi(C)^*$, and if C and D belong to M_1 (i.e., are bounded operators), then

$$\varphi(CD) = \varphi(C)\varphi(D).$$

Consequently, if C is a positive definite operator from M_1, then $\varphi(C)$ is also a positive definite operator. But then every operator D from M_1, under the mapping φ, falls into M_2, since the norm of the operator $\|D\|$ is the least number $a \geq 0$ for which the operator $a^2 I_1 - D^* D$ is positive definite. From the considerations given, it follows that $\varphi(M_1) \subset M_2$; similarly we obtain that $\varphi^{-1}(M_2) \subset M_1$. Hence, $\varphi(M_1) = M_2$.

In (^{1}) von Neumann constructed an example of a factor-representation of type II_1. From Theorem 1 it follows that all factor-representations of type II_1 for the Clifford algebra are algebraically isomorphic to this representation.

3. Let us turn to the question of the unitary classification of factor-representations of type II_1.

Let M be a factor of type II_1, whose elements act in the space H. By M', as usual, denote the commutant of the factor M. Let $f \in H$; then by $H_f^{M'}$ denote the closure of the set $M'f$ with respect to the norm of H, and by H_f^M the closure of Mf. If P_1^f is the projector onto $H_f^{M'}$, and P_2^f the projector onto H_f^M, then

$$T_M(P_1^f) = cT_{M'}(P_2^f),$$

where the constant c does not depend on f, and T_M and $T_{M'}$ are the traces for M and M', respectively. Introduce for consideration the number (2)

$$\theta = T_{M'}(I)/cT_M(I). \tag{2}$$

Under the assumption that M is a finite factor, θ varies in the range $0 < \theta \leq \infty$.

Theorem 2. *For every number θ $(0 < \theta \leq \infty)$ there exists a factor-representation of the Clifford algebra of type II_1, and this number θ determines the factor-representation uniquely up to unitary equivalence.*

For other representations of the Clifford algebra with a continuous ergodic measure (3), such a simple classification is unknown.

4. Analyzing the example of von Neumann, which was already mentioned, one can easily verify that the factor-representation of the Clifford algebra of type II_1 decomposes into a direct integral of irreducible representations for which $\nu = 1$, according to the classification of Gårding and Wightman (3). It is of interest to determine whether irreducible representations for which $\nu \neq 1$ will enter into the decomposition of a factor-representation of type II_1.

Theorem 3. *A factor-representation of type II_1 for the Clifford algebra can be decomposed into a direct integral of irreducible representations for which $\nu = 2^n$, where $n = 0, 1, 2, ...$.*

The proof of the theorem consists in constructing, for $\nu = 2^n$ $(n = 0, 1, ...)$, a factor-representation of type II_1 for the Clifford algebra that has the properties of interest to us. We note that the very method of constructing factors of type II_1 was previously unknown.

Let Γ be the space of sequences $x = x_1 x_2 ...$, where $x_i = 0$ or 1. One may regard Γ as a group if componentwise addition modulo 2 is introduced for sequences. By δ_k $(k = 1, 2, ...)$ denote the sequence from Γ with a one in the k-th place and zeros in all the others. The Haar measure μ of the group Γ coincides with Lebesgue measure under the natural mapping of Γ onto the unit interval.

Let H be the Hilbert space of vector-functions $f(x, y)$ on $\Gamma \times \Gamma$, whose values belong to a 2^n-dimensional complex prospace R_n. We define the scalar product in $\tilde{H}$ by the formula

$$\langle f, g \rangle = \int_{\Gamma\times\Gamma} (f(x,y), g(x,y))\, d\mu(x)\, d\mu(y),$$

where $(\cdot,\cdot)$ is the scalar product in R_n.

We define a representation of the anticommutation relations $\{A_k, B_k\}_1^\infty$ in $\tilde{H}$ as follows:

$$\begin{aligned} A_k f(x,y) &= j_k(x) c_k(x+y) f(x+\delta_k, y), \\ B_k f(x,y) &= i^{-1} j_{k+1}(x) c_k(x+y) f(x+\delta_k, y) \qquad (k = 1, 2, \dots), \end{aligned} \tag{3}$$

where $j_k(x) = (-1)^{x_1+\dots+x_{k-1}}$, and $\{c_k(x)\}_1^\infty$ are measurable functions on Γ, whose range consists of unitary operators in the space R_n. Similarly we define the representation $\{\tilde{A}_l, \tilde{B}_l\}_1^\infty$:

$$\begin{aligned} \tilde{A}_l f(x,y) &= j_l(y) c_l(x+y) f(x, y+\delta_l), \\ \tilde{B}_l f(x,y) &= i^{-1} j_l(y) c_l(x+y) f(x, y+\delta_l) \qquad (l = 1, 2, \dots). \end{aligned} \tag{3'}$$

Before explicitly writing the expression for $c_k(x)$ $(k = 1, 2, \dots)$, we make a remark. Denote by $\{p_s\}_1^{2n}$ an irreducible representation of the Clifford algebra in R_n with $2n$ generators $p_k = p_k^*$, $p_k^2 = 1$ $(k = 1, 2, \dots, n)$, $p_{kp}l + p_{lp}k = 0$ $(l \neq k)$ [4]. Now put

$$c_{2nk+r}(x) = s_{2nk+r}(x) p_r, \tag{4}$$

where $k = 0, 1, 2, \dots;\ 0 \le r \le 2n$,

$$s_{2nk+r}(x) = (-1)^{\sum_{i=1}^{2nk+r} x_i - \sum_{j=0}^{k} x_{2nj+r}}.$$

It is not hard to verify that the relations

$$\begin{aligned} c_k(x+\delta_k) &= c_k^*(x), \\ c_k(x) c_l(x+\delta_k) &= c_l(x) c_k(x+\delta_l) \qquad (l \neq k) \end{aligned} \tag{5}$$

are satisfied.

From the results of (3) it follows that, for almost every y, $\{A_k, B_k\}_1^\infty$ in (3) define an irreducible representation, and for almost every x, $\{\tilde{A}_k, \tilde{B}_k\}_1^\infty$ also define an irreducible representation. On the basis of Theorem 3 [6] we assert that the weakly closed ring M generated by the operators $\{A_k, B_k\}_1^\infty$, and also the ring $\tilde{M}$ generated by $\{\tilde{A}_k, \tilde{B}_k\}_1^\infty$, are factors. Moreover, it turns out that $\tilde{M} = M'$ and $M = \tilde{M}'$.

Let us verify that M and $\tilde{M}$ are factors of type II_1. For this it is enough to show [5] that on all operators from M one can define a linear homogeneous positive functional T, satisfying the conditions: 1) $T(I) = 1$, where I is the identity operator; 2) $T(AB) = T(BA)$ $(A, B \in M)$; 3) if P is a projector from M and $T(P) = 0$, then $P = 0$.

Let $e_1, \dots, e_{2^n}$ be an orthonormal basis in R_n, whose elements permute the operators $p_1, \dots, p_{2n}$. Denote $\varphi_0(x, y) \equiv 1$ and consider the vector-function $f(x, y) = \varphi_0(x, y) e_1$. Then the functional $T(A) = \langle Af, f \rangle$, where $A \in M$, has all the properties listed.

We note that if in our construction $n = 0$, then we obtain an example of von Neumann [1]. The case $n \neq 0$ has not been considered previously.

It seems interesting that if, as the measure $\mu(y)$, one takes an arbitrary measure not equivalent to Lebesgue measure, but quasi-invariant and ergodic with respect to shifts by δ_k $(k = 1, 2, \dots)$, then we obtain a factor-representation of type III for the Clifford algebra. For the case $n \neq 0$ these representations, apparently, were not known.

The author expresses sincere gratitude to Prof. M. A. Naimark for his attention to the work.

Physical-Technical Institute of Low Temperatures
Academy of Sciences of the Ukrainian SSR

Received
23 V 1966

CITED LITERATURE

[1] S. von Neumann, *Compositio Math.*, **6**, 1, 1 (1938).
[2] F. S. Murray, S. von Neumann, *Ann. Matt.*, **44**, 716 (1943).
[3] L. Gårding, A. Wightman, *Proc. Nat. Acad. Sci. U.S.A.*, **40**, No. 7, 617 (1954).
[4] T. Weyl, *Classical Groups, Their Invariants and Representations*, IL, 1947.
[5] M. A. Naimark, *Normed Rings*, M., 1956.
[6] V. Ya. Golodets, DAN, **167**, No. 1 (1966).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.