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MATHEMATICS

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ON REPRESENTATIONS OF CUBIC Z-RINGS

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1. In the present note we use the terminology and notation of the paper (1). By a cubic Z (Z_p)-ring we shall mean an order in a semisimple algebra of dimension three over the field Q (Q_p) of rational (p-adic) numbers. For such rings, B. N. Delone and D. K. Faddeev (2,3) established a connection with classes of cubic binary forms. Namely, if

$$(a, b, c, d) = ax^3 - bx^2y + cxy^2 - dy^3$$

is an arbitrary integral form with nonzero discriminant, then to it there corresponds a ring Λ with basis $[1, \omega_1, \omega_2]$ and multiplication table:

$$\begin{aligned} \omega_1\omega_2 &= \omega_2\omega_1 = ad, \\ \omega_1^2 &= b\omega_1 + a\omega_2 - ac, \\ \omega_2^2 &= d\omega_1 + c\omega_2 - bd. \end{aligned} \tag{1}$$

Conversely, in every cubic ring one can choose a basis with multiplication table (1). Isomorphic rings correspond to unimodularly equivalent forms.

If M is the maximal overring of Λ, then the factor group M/Λ is a direct sum of two cyclic groups of orders $n_1(\Lambda)$ and $n_2(\Lambda)$, where $n_1(\Lambda)$ divides $n_2(\Lambda)$.

Theorem. *A cubic ring Λ has a finite number of nonisomorphic indecomposable representations if and only if $n_1(\Lambda)$ is square-free.*

The proof is divided into several parts. All subsequent considerations are local.

2. Proposition 1. *If $M = [1, \omega_1, \omega_2]$ is a maximal cubic Z_p-ring, and $\Gamma = [1, p^2\omega_1, p^2\omega_2]$, then Γ has indecomposable representations of arbitrarily large dimension.*

Every maximal cubic Z_p-ring is isomorphic to a ring with form $(1, b, c, d)$. The only exception is (for $p = 2$) the maximal order in the algebra $Q_2 \oplus Q_2 \oplus Q_2$, having form $(0, 1, 1, 0)$.

We shall assume that the basis $[1,\omega_1,\omega_2]$ corresponds to the form $(1,b,c,d)$. Let A be a Γ-module with basis $[1,\omega_1,p\omega_2]$, and let Ω_1 (Ω_2) be the matrix of the operator $p^2\omega_1$ $(p^2\omega_2)$ in the module A,

$$\alpha_1=\begin{pmatrix}0 & pc & -p^2d\\ p & 0 & 0\\ 0 & 0 & 0\end{pmatrix};\qquad \alpha_2=\begin{pmatrix}0 & -pd & p^2bd\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix};$$

$$\beta_1=\begin{pmatrix}-pc & 0 & 0\\ pb & pc & p^2d\\ 1 & 0 & 0\end{pmatrix};\qquad \beta_2=\begin{pmatrix}pd & 0 & 0\\ 0 & -pd & p^2d\\ 0 & 0 & 0\end{pmatrix};$$

E is the identity matrix of size $k\times k$, J is a degenerate Jordan block of the same size

$$J=\begin{pmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & \cdots & 1\\ 0 & 0 & 0 & \cdots & 0\end{pmatrix};$$

$\otimes$ is the sign of tensor multiplication of matrices. Then the correspondence

$$p^2\omega_i\to\begin{pmatrix}\Omega_i\otimes E & \alpha_i\otimes E+\beta_i\otimes J\\ 0 & \Omega_i\otimes E\end{pmatrix}\qquad (i=1,2)$$

is a representation of the ring Γ, whose decomposition contains an indecomposable piece of dimension at least $2k$. An analogous result holds also for $p=2$ for the ring M with form $(0,1,1,0)$.

3. In what follows we shall use the same notation for a representation and its module. If $A_1,\dots,A_s$ are representation modules, then the module B will be called of type $(A_1,\dots,A_s)$ if B can be brought to the form:

$$B=\begin{pmatrix}A_1 & & * & \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_s\end{pmatrix}.$$

For rings Λ not contained in Γ (upper rings), one can prove the finiteness of the number of nonisomorphic indecomposable representations and describe these representations. Obviously, it suffices to restrict oneself to self-nonconjugate rings $(\Lambda\not\approx\Lambda^*)$, since if $\Lambda\approx\Lambda^*$, then every exact indecomposable module is invertible and therefore coincides with Λ (1). The result depends on the type of the algebra $\hat{\Lambda}=\mathfrak{A}$.

a) $\mathfrak{A}$ is an unramified field. In the maximal order M one can choose a basis $[1, \omega_1, \omega_2]$ with form $(\varepsilon, 0, c, 1)$, where the polynomial $\varepsilon t^3 + ct - 1$ has no roots mod p, and ε is not divisible by p. In $\mathfrak{A}$ there is one self-nonconjugate upper order $\Lambda = [1, p\omega_1, p\omega_2]$.

Proposition 2. Λ has 4 nonisomorphic indecomposable representations: Λ, Λ^*, M, and one representation of type (Λ, Λ^*).

b) $\mathfrak{A}$ is a ramified field. In the maximal order M one can choose a basis $[1, \omega_1, \omega_2]$ with form $(\varepsilon, 0, c, p)$ (c divisible by p). The exceptions (for $p = 3$) are rings with forms $(\varepsilon, 0, 3, 1)$ ($\varepsilon \equiv -1 \pmod 3$, $\varepsilon \not\equiv 5 \pmod 9$). In $\mathfrak{A}$ there are two self-nonconjugate upper orders: $\Lambda = [1, p\omega_1, p\omega_2]$, $\Sigma = [1, p^2\omega_1, p\omega_2]$.

Proposition 3. Λ has 6 indecomposable representations: Λ, Λ^*, $K = [1, p\omega_1, \omega_2]$, M, one representation of type (Λ, Λ^*), and one of type (Λ, M). The ring Σ has 18 indecomposable representations: those listed above, and also Σ, Σ^*, one representation of type (Σ, K), two of type (Σ, M), one of type (Σ, Σ^*), one of type (Σ, Λ^*), one of type $(\Sigma \oplus \Sigma, M)$, one of type $(\Sigma, M \oplus \Lambda^*)$, one of type $(\Sigma \oplus \Lambda, \Lambda^*)$, one of type $(\Sigma \otimes \Sigma, M \oplus \Lambda^*)$, and one of type $(\Sigma \otimes \Lambda, M \oplus \Lambda^*)$.

An analogous result holds (for $p = 3$) also for subrings of the maximal ring with form $(\varepsilon, 0, 3, 1)$.

c) $\mathfrak{A} = Q_p \oplus Q_p \oplus Q_p$. Let e_1, e_2, e_3 be a system of minimal idempotents in $\mathfrak{A}$. Then every upper self-nonconjugate ring is isomorphic to one of the following:

$$\Lambda_k = [1, pe_2, p^k e_3] \qquad (k > 0),$$

$$L_{k,\theta} = [1, pe_2 + \theta p^{k-1} e_3, p^k e_3] \qquad (k > 1,\ \theta \text{ is a unit in } Z_p).$$

Irreducible—

decomposable Λ-modules are A_i ($i = 1, 2, 3$), where A_i is a free Z_p-module of rank 1 with generators a_i, $a_i e_j = \delta_{ij} a_i$.

Proposition 4. The ring $L_{k,\theta}$ has $4k + 1$ indecomposable representations: A_1, A_2, A_3, one representation U of type (A_1, A_2), one V of type (A_2, A_3), $k-1$ of type (A_1, A_3), k of type (U, A_3), $k-1$ of type (A_1, V), $k-2$ of type (U, V). The ring Λ_k has $4(k+1)$ indecomposable representations (one more representation of each of the types (A_1, A_3), (A_1, V), (U, V) is added).

c) $\mathfrak{A} = Q_p \oplus \mathfrak{A}_1$ ($\mathfrak{A}_1 = Q_p(\sqrt{\alpha})$, $\alpha \in Z_p$ and is not divisible by p^2). Let e_1, e_2 be orthogonal idempotents in $\mathfrak{A}$, $\dim_{Q_p}(e_1\mathfrak{A}) = 1$. A maximal order in $\mathfrak{A}_1$ has basis $[e_2, \omega]$, $\omega^2 = \alpha e_2$ (except for the case $p = 2$, $\alpha = -3$). The upper nonconjugate orders in $\mathfrak{A}$ have the form

$$\Lambda_k = [1, p^k\omega, pe_1] \qquad (k > 0),$$

$$L_{k,\theta} = [1, p^k\omega + \theta pe_1,\ p^2e_1] \qquad (k > 0,\ \theta \text{ a unit in } Z_p), \quad K = [1, p\omega, p^2e_1]$$

(only when α is divisible by p).

The rings Λ_k and $L_{k,\theta}$ have one irreducible representation A, belonging to the first component, and $k+1$ representations belonging to the second component: $U_0, U_1, \dots, U_k$.

Proposition 5. If α is not divisible by p, then the ring Λ_k has $4k+2$ indecomposable representations: A, U_i $(i = 0, \dots, k)$, one of type (U_0, A), two each of types (U_i, A) $(i = 1, \dots, k-1)$, one of type (U_k, A), and one each of types $(U_i, A \oplus A)$ $(i = 0, \dots, k-1)$. The ring $L_{k,\theta}$ has $4k+3$ indecomposable representations (to the k listed above one more of type (U_k, A) is added).

Proposition $5'$**.** If α is divisible by p, then the ring Λ_k $(L_{k,\theta})$ has $4k+3$ (respectively, $4(k+1)$) indecomposable representations (to the k listed above one more of type (U_0, A) is added). The ring K has 12 indecomposable representations: A, U_0, U_1, three of type (U_0, A), two of type (U_1, A), two of type $(U_0, A \oplus A)$, one of type $(U_0 \oplus U_1, A)$, and one of type $(U_0 \oplus U_1, A \oplus A)$.

An analogous result also holds for $p = 2$, $\mathfrak{A}_1 = Q_p(\sqrt{-3})$. From Propositions 3-5, $5'$ it follows that all upper cubic Z_p-rings have a finite number of nonisomorphic indecomposable representations. The globalization of this result gives the main theorem formulated in § 1.

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