SOLUTION OF EQUATIONS IN WORDS WITH THREE UNKNOWNS

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SOLUTION OF EQUATIONS IN WORDS WITH THREE UNKNOWNS

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The present paper is devoted to the solution of systems of equations in words with three unknowns. Equations in words were considered in the works ([1,2]). In ([1]), W_j-systems of equations in words were defined, and it was noted that the solution of an arbitrary system of equations reduces to the solution of W_3-systems. It was asserted there that there exists an algorithm which, for any W_2-system of equations, determines whether this system has a solution or not. In ([2]), equations in words with three unknowns without coefficients were considered, and it was asserted that the solutions of such equations are representable by parametric words.

After the solution of W_2-systems, the natural next step is to consider systems of equations in words with three unknowns. In view of Theorem 2 of ([1]), it suffices to consider one equation with three unknowns. We shall set forth the main ideas of the solution of equations in words with three unknowns and formulate the theorems from which the deciding algorithm follows.

First of all, we restrict ourselves to equations of the form $x\Phi = y\Psi$, i.e., equations whose left- and right-hand sides begin with different unknowns (the general case is easily reduced to this one). Further, it turns out to be convenient to introduce into consideration "one-sided" equations, which we denote by the symbol $x\Phi \to y\Psi$. A **solution of the equation** $x\Phi \to y\Psi$ is any solution $(X, Y, \ldots)$ of the equation $x\Phi = y\Psi$ such that $|X^\partial| \geq |Y^\partial| > 0$. Obviously, it is sufficient to construct an algorithm for solving one-sided equations. Multiplying, if necessary, the equation $x\Phi \to y\Psi$ on the right by x, we write it in the form*

$$x\Phi \to [\xi_i A_i]_{i=1}^k x\Psi, \tag{1}$$

where $k \geq 1$; $\xi_1, \ldots, \xi_k$ are unknowns distinct from x; $A_1, \ldots, A_k$ are coefficients. The basic arguments and constructions pertain to equations of the form (1). We give the necessary definitions as applied to the general case of an equation with n unknowns. We assume that x is a letter from the list $x_1, \ldots, x_n$.

Define finite and infinite images of equation (1). Every equality of the form

$$x = ([\xi_i A_i]_{i=1}^k)^t [\xi_i A_i]_{i=1}^r \, \xi_{r+1} P, \qquad (2)$$

where $t \geq 0$; $0 \leq r < k$, and P is some beginning of the word A_{r+1}, will be called a **finite transformation** of equation (1). The equation

$$S^x_{([\xi_i A_i]_1^k)^t [\xi_i A_i]_1^r \xi_{r+1} P} \, x\Phi = S^x_{([\xi_i A_i]_1^{kt})[\xi_i A_i]_1^r \xi_{r+1} P} [\xi_i A_i]_1^k x \Psi \qquad (2')$$

will be called the **finite image** of equation (1) under the transformation (2).

Every equality of the form

$$x = ([\xi_i A_i]_{i=1}^k)^t [\xi_i A_i]_{i=1}^r y, \qquad (3)$$

where $t \geq 0$; $0 \leq r < k$; $kt + r > 0$; y is an unknown distinct from $x_1, \dots, x_n$,

* The symbol $[P_i]_{i=m}^n$ denotes the word $P_{m P_{m+1}} \dots P_n$ if $m \leq n$, and the empty word if $m > n$.

we shall call a **nonfinite transformation** of equation (1). The equation

$$y S_{\{[\xi_i A_i]_1^k\}^x \{\xi_i A_i\}_1^r y} \Phi \leftarrow [\xi_i A_i]_{i=r+1}^k [\xi_i A_i]_{i=1}^r y S_{\{[\xi_i A_i]_1^k\}^x \{\xi_i A_i\}_1^r y} \Psi \qquad (3')$$

we shall call a **nonfinite image of equation** (1) **under the transformation** (3).

Every finite image of an equation with n unknowns is an equation with $n-1$ unknowns; nonfinite images of equations with n unknowns are equations with n unknowns.

Denote equation (1) by $\mathfrak{A}$, equation $(2')$ by $\mathfrak{A}_{t,r,P}$, and equation $(3')$ by $\mathfrak{A}''_{t,r}$. The following lemma reduces the solution of equation (1) to the solution of images of this equation.

Lemma. *The following equivalence holds*

$$\mathfrak{A}(x_1, \dots, x_n) \Longleftrightarrow \bigvee_{r=0}^{k-1} \bigvee_{P < \Delta_{r+1}} \exists t \, [x = ([\xi_i A_i]_1^k)^t [\xi_i A_i]_1^r \xi_{r+1} P \& \mathfrak{A}'_{t,r,P}]$$

$$\vee \bigvee_{r=0}^{k-1} \exists t \exists y \, [x = ([\xi_i A_i]_1^k)^t [\xi_i A_i]_1^r y \& \mathfrak{A}''_{t,r}].$$

According to the indicated equivalence, solving the equation $\mathfrak{A}$ is reduced to solving the images of this equation, for which it is necessary to review an infinite number of images. However, in the case of equations with three unknowns one can restrict oneself to the consideration of a finite number of images. This is done in the following way.

Consider the equality

$$\Phi(x_1, \dots, x_n, t) = \Psi(x_1, \dots, x_n, t), \tag{4}$$

where Φ, Ψ are terms in the sense of (1), t is a natural unknown, and $x_1, \dots, x_n$ are W-unknowns. Equation (4) is an equation with $n+1$ unknowns $x_1, \dots, x_n, t$. Substituting natural numbers for t in (4), we obtain equations in words with n unknowns $x_1, \dots, x_n$. We say that the parameter t is **eliminable in equation** (4) if there exist natural numbers $c_1, \dots, c_m$ such that the equivalence holds

$$\exists x_1 \dots x_n t(\Phi = \Psi) \Longleftrightarrow$$

$$\Longleftrightarrow \bigvee_{i=1}^{m} \exists x_1 \dots x_n [\Phi(x_1, \dots, x_n, c_i) = \Psi(x_1, \dots, x_n, c_i)].$$

The **eliminability of a parameter** for one-sided equations is defined analogously. According to this definition, if the parameter t is eliminable in equation (4), then the question of the solvability of equation (4), considered as an equation with $n+1$ unknowns, is reduced to the question of the solvability of a finite number of equations in words with n unknowns.

We say that the parameter is eliminable in the images of equation (1) if the parameter t is eliminable in equations $(2')$, $(3')$ for all r, P.

Theorem 1. *The parameter is eliminable in images of equations in words with three unknowns.*

Theorem 1 makes it possible to restrict oneself to the consideration of a finite number of images of equation (1). Moreover, only nonfinite images are of interest, since finite images are equations with two unknowns, and for the latter there is a decision procedure (see [1]). To solve images it is sufficient to consider images of these images, and so on. A sequence $\mathfrak{A}_0, \dots, \mathfrak{A}_k$ ($k \geqslant 0$) of equations in words will be called a **nonfinite chain** (the equation $\mathfrak{A}_0$), if for each $1 \leqslant i \leqslant k$ the equation $\mathfrak{A}_i$ is a nonfinite image of the equation $\mathfrak{A}_{i-1}$. A sequence $\mathfrak{A}_0, \dots, \mathfrak{A}_k$ ($k \geqslant 1$) of equations in words will be called a **finite chain** if

$\mathfrak{A}_0, \dots, \mathfrak{A}_{k-1}$ is a nonfinitary chain, and $\mathfrak{A}_k$ is a finitary image of the equation $\mathfrak{A}_{k-1}$. Our method of considering equations in words consists in studying the nonfinitary chains of these equations. More precisely, we are interested in the question of the behavior of the equation $\mathfrak{A}_k$ in the nonfinitary chain $\mathfrak{A}_0, \dots, \mathfrak{A}_k$, as $k \to \infty$. It turns out that, for equations with three unknowns, the equation $\mathfrak{A}_k$, as k grows, either assumes an increasingly special form, or else takes previously assumed values with a known periodicity, or there is an alternation of these phenomena. These regularities are reflected in the following theorems.

Theorem 2. For every equation $\mathfrak{A}_0$ with three unknowns there exists $l(\mathfrak{A}_0)$ such that every chain $\mathfrak{A}_0, \dots, \mathfrak{A}_l$ contains an equation of one of the following forms (to shorten the notation we omit coefficients)

$$xy^{az}\Phi \to zy^{bx}\Psi \qquad (a, b \geq 0); \tag{5}$$

$$xyz\Phi \to zxy\Psi; \tag{6}$$

$$xy^{zzx}\Phi \to zx^2y^2\Psi; \tag{7}$$

$$xy^2\alpha\Phi \to zyx\Psi \qquad (\alpha \neq y). \tag{8}$$

Theorem 3. The decidability problem for equations of the forms (5)–(8) reduces to the decidability problem for equations of the following two forms:

$$xAy\Phi \to yBx\Psi \qquad ([A^\partial] = [B^\partial]); \tag{9}$$

$$xAyBz\Phi \to zCyDx\Psi \qquad ([AB^\partial] = [CD^\partial]), \tag{10}$$

where x, y, z are unknowns; A, B, C, D are coefficients; Φ, Ψ are words in the alphabet $\Pi \cup \{x, y, z\}$.

Theorem 4. The following alternative holds: 1) equation (10) is equivalent to the equation $xAyBz \to zCyDx$; 2) there exists $l(\Phi, \Psi)$ such that every chain $\mathfrak{A}_0, \dots, \mathfrak{A}_l$ of equation (10) contains an equation equivalent to an equation of the form (9).

Remark. A decision algorithm for the equation $xAyBz \to zCyDx$ is constructed quite simply. (An algorithm is also constructed quite simply for analogous equations with n unknowns of the form $x_1A_1 \dots x_{nA}n = x_{i_1}B_1 \dots x_{i_n}B_n$, where A_i, B_i are coefficients and $(i_1, \dots, i_n)$ is some permutation of the elements $1, \dots, n$.)

Theorem 2 is proved by a direct count of images in chains. In the proof of Theorems 3 and 4, auxiliary considerations are used: properties of the function F from (1), conditions for equivalence of equations, a decision algorithm for equations with two unknowns, etc.

After Theorems 1, 2, 3, and 4, it remains to solve equation (9). The peculiarity of equation (9) is that its transformations do not affect the unknown z. On the other hand, here a direct method applies: we find the general solution of the equation $xAy \to yBx$ and substitute it into equation (9). Then we eliminate (at least in one part of equation (9)) the prefix consisting of the unknowns x, y that blocks access to z. Elimination of the prefix makes it possible to find a relation between z and the formulas for x, y. Analysis of this relation constitutes the final stage in solving equation (9). Here an essential role is played by the methods used in solving W_2-systems. The corresponding theorem is formulated in the same way as Theorem 1 from (1), with the difference that instead of W_2-systems equation (9) now appears, and instead of the unknowns $x_1, \dots, x_n$, the unknowns x, y, z appear.

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[1] Yu. I. Khmelevskii, DAN, **156**, No. 4 (1964). [2] Yu. I. Khmelevskii, DAN, **171**, No. 5 (1966).

Note: Figure translations are in progress. See original paper for figures.